MATH 555 NOTES
Examples of Functions which are Riemann Integrable

Throughout this section, we consider a and b to be real numbers with $a < b$. Our goal is to develop some theorems which will easily lead to examples of Riemann integrable functions. We begin with a criterion for Riemann integrability attributed to Riemann.

Theorem. *Let $f : [a, b] \mapsto \mathbb{R}$ be bounded. Then $f \in \mathcal{R}[a, b]$ if and only if for every $\varepsilon > 0$ there is a partition $\mathcal{P}$ of $[a, b]$ such that $U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon$.*

Lemma. *Let $f : [a, b] \mapsto \mathbb{R}$ be bounded. Then $\underline{\int_a^b} f \le \overline{\int_a^b} f$.*

Proof. Use the definition of $\underline{\int_a^b} f$ and $\overline{\int_a^b} f$ together with $L(\mathcal{P}, f) \le L(\mathcal{P} \cup \mathcal{Q}, f) \le U(\mathcal{P} \cup \mathcal{Q}, f) \le U(\mathcal{Q}, f)$ for all partitions $\mathcal{P}$ and $\mathcal{Q}$ of $[a, b]$. ■

Proof of Theorem. Suppose first that $f \in \mathcal{R}[a, b]$. Let $\varepsilon > 0$. Since $\int_a^b f = \overline{\int_a^b} f = \underline{\int_a^b} f$, we obtain that there are partitions $\mathcal{P}_1$ and $\mathcal{P}_2$, such that

$$U(\mathcal{P}_1, f) < \int_a^b f + \frac{\varepsilon}{2} \quad \text{and} \quad L(\mathcal{P}_2, f) > \int_a^b f - \frac{\varepsilon}{2}.$$

If $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, then by the theorem from the previous section we deduce

$$U(\mathcal{P}, f) \le U(\mathcal{P}_1, f) < \int_a^b f + \frac{\varepsilon}{2} = \int_a^b f - \frac{\varepsilon}{2} + \varepsilon < L(\mathcal{P}_2, f) + \varepsilon \le L(\mathcal{P}, f) + \varepsilon,$$

establishing the inequality in the theorem.

Now, suppose we know for every $\varepsilon > 0$ there is a partition $\mathcal{P}$ of $[a, b]$ such that $U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon$, and we want to show $f \in \mathcal{R}[a, b]$. Fix any such ε and $\mathcal{P}$. By the Lemma,

$$\underline{\int_a^b} f \le \overline{\int_a^b} f \le U(\mathcal{P}, f) < L(\mathcal{P}, f) + \varepsilon \le \underline{\int_a^b} f + \varepsilon.$$

Since $\underline{\int_a^b} f \le \overline{\int_a^b} f < \underline{\int_a^b} f + \varepsilon$ for every $\varepsilon > 0$, we deduce $\overline{\int_a^b} f = \underline{\int_a^b} f$ which implies that $f \in \mathcal{R}[a, b]$. ■

Questions: We have shown that $U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon$ holds for some partition $\mathcal{P}$ of $[a, b]$; must it hold for infinitely many partitions of $[a, b]$? Why? How might the above theorem be used?

Theorem. *Let $f : [a, b] \mapsto \mathbb{R}$ be monotone. Then $f \in \mathcal{R}[a, b]$.*

Proof. Suppose f is monotone increasing. Let $\varepsilon > 0$. Consider a positive integer $n > (b-a)(f(b) - f(a))/\varepsilon$, and let $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ where $x_j = a + j(b-a)/n$. Thus, $x_j - x_{j-1} = (b-a)/n$ for each $j \in \{1, 2, \ldots, n\}$. Since f is increasing,

$$\begin{aligned} U(\mathcal{P}, f) - L(\mathcal{P}, f) &= \sum_{j=1}^{n} f(x_j)(x_j - x_{j-1}) - \sum_{j=1}^{n} f(x_{j-1})(x_j - x_{j-1}) \\ &= \sum_{j=1}^{n} \big(f(x_j) - f(x_{j-1})\big)(x_j - x_{j-1}) \\ &= \frac{b-a}{n} \sum_{j=1}^{n} \big(f(x_j) - f(x_{j-1})\big) = \frac{b-a}{n}\big(f(b) - f(a)\big) < \varepsilon. \end{aligned}$$

The result for f increasing now follows from the previous theorem. A similar argument works if f is monotone decreasing. ■

Theorem. *Let $f : [a, b] \mapsto \mathbb{R}$ be continuous. Then $f \in \mathcal{R}[a, b]$.*

Proof. Let $\varepsilon > 0$. Since f is continuous on the compact set $[a, b]$, it is uniformly continuous on $[a, b]$. Thus, there is a $\delta > 0$ such that if $|x - y| < \delta$ and x and y are in $[a, b]$ then $|f(x) - f(y)| < \varepsilon/(2(b-a))$. Let $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a, b]$ with $\|\mathcal{P}\| < \delta$. It follows that for each $j \in \{1, 2, \ldots, n\}$, $0 \le M_j(f) - m_j(f) \le \varepsilon/(2(b-a))$. Thus,

$$0 \le U(\mathcal{P}, f) - L(\mathcal{P}, f) = \sum_{j=1}^{n} \big(M_j(f) - m_j(f)\big)(x_j - x_{j-1}) \le \frac{\varepsilon}{2(b-a)} \sum_{j=1}^{n} (x_j - x_{j-1}) < \varepsilon.$$

It follows from the first theorem in this section that f is Riemann integrable on $[a, b]$. ■

Questions: What are some examples of Riemann integrable functions? Is the function $f(x)$ defined by $f(x) = x$ for $-1 \le x \le 0$ and by $f(x) = x^2$ for $0 < x \le 1$ in $\mathcal{R}[-1, 1]$? Is the function $f(x)$ defined by $f(x) = x^2$ for $-1 \le x \le 0$ and by $f(x) = x + 1$ for $0 < x \le 1$ in $\mathcal{R}[-1, 1]$? How might the proof of the last theorem be modified to take care of a discontinuity?

Homework:

(1) Let $f : [a, b] \mapsto \mathbb{R}$ be such that f is bounded and has a finite number of discontinuities. Prove that $f \in \mathcal{R}[a, b]$.

(2) Let $f : [0, 1] \mapsto \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x = 1/n \text{ for some } n \in \mathbb{Z}^+ \\ x^2 & \text{otherwise.} \end{cases}$$

Determine whether f is Riemann integrable on $[0, 1]$.

(3) (a) Suppose $f \in \mathcal{R}[a, b]$. Prove that $f^2 \in \mathcal{R}[a, b]$.
(b) Suppose $f^2 \in \mathcal{R}[a, b]$. Explain why it may not be the case that $f \in \mathcal{R}[a, b]$.