MATH 555 NOTES
The Definition of the Riemann Integral

Definitions and Notations. Let $a < b$. A *partition* of $[a, b]$ is a finite set $\{x_0, x_1, \ldots, x_n\}$ satisfying $x_0 = a$, $x_n = b$, and $x_0 < x_1 < \cdots < x_n$. Suppose $f : [a, b] \mapsto \mathbb{R}$ is bounded, and $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$. For $j \in \{1, 2, \ldots, n\}$, let

$$M_j(f) = \sup\{f(t) : t \in [x_{j-1}, x_j]\} \quad \text{and} \quad m_j(f) = \inf\{f(t) : t \in [x_{j-1}, x_j]\}.$$

Define the *upper sum* and *lower sum* for f by

$$U(\mathcal{P}, f) = \sum_{j=1}^{n} M_j(f)(x_j - x_{j-1}) \quad \text{and} \quad L(\mathcal{P}, f) = \sum_{j=1}^{n} m_j(f)(x_j - x_{j-1}),$$

respectively. Define the *upper integral* and *lower integral* of f by

$$\overline{\int_a^b} f = \inf\{U(\mathcal{P}, f) : \mathcal{P} \text{ is a partition of } [a, b]\}$$

and

$$\underline{\int_a^b} f = \sup\{L(\mathcal{P}, f) : \mathcal{P} \text{ is a partition of } [a, b]\},$$

respectively. Then we say that f *is Riemann integrable on* $[a, b]$ if $\overline{\int_a^b} f = \underline{\int_a^b} f$. We call this common value of the upper and lower integrals the *Riemann integral of f on* $[a, b]$, and we denote it by $\int_a^b f$ or $\int_a^b f(x)\, dx$. We let $\mathcal{R}[a, b]$ denote the set of all Riemann integrable functions on $[a, b]$.

Questions: In the definitions of $M_j(f)$ and $m_j(f)$ above, why were "sup" and "inf" used instead of "max" and "min"? Why do the upper and lower integrals exist above (why are they finite)? What is this definition saying geometrically? In particular, if $f > 0$ on $[a, b]$, what does the Riemann integral of f on $[a, b]$ represent geometrically? What relationship exists between $U(\mathcal{P}, f)$ and $L(\mathcal{P}, f)$?

Defintion. If $\mathcal{P}$ and $\mathcal{Q}$ are partitions of $[a, b]$ with $\mathcal{P} \subseteq \mathcal{Q}$, then $\mathcal{Q}$ is called a *refinement* of $\mathcal{P}$.

Theorem. *Let f be as above. If $\mathcal{P}$ and $\mathcal{Q}$ are partitions of $[a, b]$ with $\mathcal{Q}$ a refinement of $\mathcal{P}$, then*

$$L(\mathcal{P}, f) \le L(\mathcal{Q}, f) \le U(\mathcal{Q}, f) \le U(\mathcal{P}, f).$$

Proof. Use that if $y \in [x_{j-1}, x_j]$, then

$$\sup\{f(t) : t \in [x_{j-1}, x_j]\} = \max\{\sup\{f(t) : t \in [x_{j-1}, y]\}, \sup\{f(t) : t \in [y, x_j]\}\}$$

and

$$\inf\{f(t) : t \in [x_{j-1}, x_j]\} = \min\{\inf\{f(t) : t \in [x_{j-1}, y]\}, \inf\{f(t) : t \in [y, x_j]\}\}$$

(but note only inequalities are needed). From these it easily follows that if $\mathcal{Q}$ contains only one more element than $\mathcal{P}$, then the inequalities in the theorem hold. For general $\mathcal{Q}$, one can use induction. ■

Defintion and Notation. If $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, b]$, then the *mesh of the paritition* $\mathcal{P}$ is defined to be $\|P\| = \max\limits_{1 \leq j \leq n} \{x_j - x_{j-1}\}$.

Questions: What significance might the mesh of a partition have in connection to the integral of f? Does Problem (1) below shed any light on this?

Homework:

(1) Let f be as above. Prove that for every $\varepsilon > 0$,

$$\begin{aligned}&\inf\{U(\mathcal{P}, f) : \mathcal{P} \text{ is a partition of } [a, b]\}\\ &\qquad = \inf\{U(\mathcal{P}, f) : \mathcal{P} \text{ is a partition of } [a, b] \text{ and } \|P\| < \varepsilon\}\end{aligned}$$

and

$$\begin{aligned}&\sup\{L(\mathcal{P}, f) : \mathcal{P} \text{ is a partition of } [a, b]\}\\ &\qquad = \sup\{L(\mathcal{P}, f) : \mathcal{P} \text{ is a partition of } [a, b] \text{ and } \|P\| < \varepsilon\}.\end{aligned}$$

(2) Define $f : [0, 1] \mapsto \mathbb{R}$ by $f(x) = 1$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \notin \mathbb{Q}$. Prove that f is not Riemann integrable on $[0, 1]$.

(3) Define $f : [0, 1] \mapsto \mathbb{R}$ by $f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = x$ if $x \notin \mathbb{Q}$. Compute $\overline{\int_0^1} f$ and $\underline{\int_0^1} f$. Is f Riemann integrable on $[0, 1]$?

(4) Let a and b be real numbers with $a < b$. Prove that if $f(x) = c$ for all $x \in [a, b]$ where c is a constant, then $\int_a^b f(x)\,dx = c(b - a)$.

(5) Let a and b be real numbers with $a < b$. Using only material from this section, explain why $f(x) = x$ is Riemann integrable on $[a, b]$ and determine (with proof) the value of $\int_a^b x\,dx$.

(6) Let f and g be elements of $\mathcal{R}[a, b]$ with $f(x) \leq g(x)$ for all $x \in [a, b]$. Prove that $\int_a^b f \leq \int_a^b g$.