MATH 555 NOTES
The Cauchy Mean Value Theorem and L'Hôpital's Rule

Theorem (The Cauchy Mean Value Theorem). *Let a and b be real numbers with $a < b$, and let $f : [a, b] \mapsto \mathbb{R}$ and $g : [a, b] \mapsto \mathbb{R}$ with both f and g continuous on $[a, b]$ and differentiable on (a, b). Then there is an $x_0 \in (a, b)$ for which*

$$\big(f(b) - f(a)\big)g'(x_0) = \big(g(b) - g(a)\big)f'(x_0).$$

Proof. Let

$$h(x) = \big(f(b) - f(a)\big)g(x) - \big(g(b) - g(a)\big)f(x),$$

and note that

$$h(b) - h(a) = \big(f(b) - f(a)\big)\big(g(b) - g(a)\big) - \big(g(b) - g(a)\big)\big(f(b) - f(a)\big) = 0.$$

Apply the Mean Value Theorem to get the result. ■

Questions: How is this result related to Rolle's Theorem and the Mean Value Theorem? What is the Cauchy Mean Value Theorem saying geometrically (think in terms of the parametric equations $y = f(t)$ and $x = g(t)$)?

Theorem (L'Hôpital's Rule). *Let a and b be real numbers with $a < b$, and let $f : [a, b] \mapsto \mathbb{R}$ and $g : [a, b] \mapsto \mathbb{R}$ with both f and g continuous on $[a, b]$ and differentiable on (a, b). Suppose $g'(x) \neq 0$ for $x \in (a, b)$. Suppose further that $x_0 \in [a, b]$ is such that*

$$f(x_0) = g(x_0) = 0 \quad \text{and} \quad \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \text{ exists.}$$

Then $\lim_{x \to x_0} f(x)/g(x)$ exists and

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

Proof. First, observe that Rolle's Theorem and the conditions $g(x_0) = 0$ and $g'(x) \neq 0$ for $x \in (a, b)$ imply that $g(x) \neq 0$ for all $x \in [a, b] - \{x_0\}$. Let $\{x_n\}_{n=1}^{\infty}$ be an arbitrary sequence converging to x_0 with $x_n \in [a, b] - \{x_0\}$. In particular, we have $g(x_n) \neq 0$ for each $n \geq 1$. By the Cauchy Mean Value Theorem, for each $n \geq 1$, there is a t_n between x_0 and x_n for which

$$(*) \qquad \frac{f(x_n)}{g(x_n)} = \frac{f(x_n) - f(x_0)}{g(x_n) - g(x_0)} = \frac{f'(t_n)}{g'(t_n)}.$$

Since $x_n \to x_0$, we deduce $t_n \to x_0$. Hence, the right-hand side of $(*)$ approaches a limit as $n \to \infty$, namely the value of $\lim_{x \to x_0} f'(x)/g'(x)$. We deduce that the left-hand side of $(*)$ also approaches this limit as $n \to \infty$. The result follows. ■

Questions: What is the value of $\lim_{x \to 0} \frac{x}{e^x - 1}$? What is the value of $\lim_{x \to 0} \frac{\sin x}{x}$? Why would a student evaluating this limit using L'Hôpital's Rule make a mathematics teacher sick? What other variations of L'Hôpital's Rule do you remember from Calculus? Can we apply the version established above to $\lim_{x \to 0^+} e^{-1/x}/x$? Can we apply a different variation of L'Hôpital's Rule for this limit (be creative)?

Homework:

(1) Explain how L'Hôpital's Rule as we have established it above implies $\lim_{x \to 1} x^{1/(x-1)}$ exists. Also, determine the value of this limit. (You may use rules for differentiating from Calculus which have not been discussed previously in this course.)

(2) Repeat exercise (1) for $\lim_{x \to 0} \frac{x - \sin x}{x^3}$. Be careful how you explain the application of L'Hôpital's Rule.