MATH 555 NOTES
The Mean Value Theorem

Definition. Let $x_0 \in E \subseteq \mathbb{R}$, and let $f : E \mapsto \mathbb{R}$. If $f(x) \le f(x_0)$ ($f(x) \ge f(x_0)$) for all $x \in E$, then f *has an absolute maximum (absolute minimum) at* x_0 *on* E. The absolute maximum (absolute minimum) is $f(x_0)$. If there is an $\varepsilon > 0$ such that f obtains an absolute maximum (absolute minimum) at x_0 on $E \cap N_\varepsilon(x_0)$, then f *has a local maximum (local minimum) at* x_0. In this case, $f(x_0)$ will be a local maximum (local minimum).

Comment: Don't expect consistency in the literature here. Some books are silly enough to refer to x_0 above rather than $f(x_0)$ as the absolute maximum. Other terminology includes the use of "global" instead of "absolute" above and the use of "relative" instead of "local".

Questions: What from Calculus do you remember about local maximums or local minimums of a function f and the derivative f'? What are the local maximums and local minimums for $f(x) = x$ on $[0, 1]$, for $f(x) = |x|$ on $(-1, 1)$, and for $f(x) = x^2$ on $[-1, 1]$? Are absolute maximums local maximums? Are local minimums absolute minimums? Can an absolute maximum be equal to a local minimum? Can a non-constant function have this property? If f has a local maximum at x_0, what can you say about $-f$ at x_0? Suppose $E = [a, b]$; what condition on f would imply that f has an absolute maximum and an absolute minimum at some points on E?

Theorem 1. *Let a and b be real numbers with $a < b$, and let $f : [a, b] \mapsto \mathbb{R}$. Suppose f has a local maximum or local minimum at $x_0 \in (a, b)$. If f is differentiable at x_0, then $f'(x_0) = 0$.*

Proof. Suppose f has a local maximum at x_0 (the argument for a local minimum is similar). Let $\varepsilon > 0$. There is a $\delta > 0$ such that $N_\delta(x_0) \subseteq [a, b]$ and if $0 < |x - x_0| < \delta$ then

$$\left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \varepsilon.$$

It follows that

$$(*) \quad \frac{f(x) - f(x_0)}{x - x_0} - \varepsilon < f'(x_0) < \frac{f(x) - f(x_0)}{x - x_0} + \varepsilon \quad \text{for } x \in (x_0 - \delta, x_0 + \delta) \text{ with } x \ne x_0.$$

Since f has a local maximum at x_0, we deduce that $f(x) - f(x_0) \le 0$ for x in some neighborhood of x_0. By considering the first inequality in $(*)$ for some $x < x_0$, we deduce that $f'(x_0) > -\varepsilon$. By considering the second inequality in $(*)$ for some $x > x_0$, we deduce that $f'(x_0) < \varepsilon$. Since $-\varepsilon < f'(x_0) < \varepsilon$ holds for every $\varepsilon > 0$, we obtain $f'(x_0) = 0$. ∎

Theorem 2 (Rolle's Theorem). *Let a and b be real numbers with $a < b$, and let $f : [a, b] \mapsto \mathbb{R}$ with f continuous on $[a, b]$ and f differentiable on (a, b). If $f(a) = f(b) = 0$, then there is an $x_0 \in (a, b)$ for which $f'(x_0) = 0$.*

Proof. Since f is continuous on $[a, b]$ (a compact set), there exist x_1 and x_2 in $[a, b]$ with $f(x_1)$ the absolute minimum and $f(x_2)$ the absolute maximum of f on $[a, b]$. If either of x_1 or x_2 is in (a, b), then Theorem 1 implies Theorem 2 holds. If neither x_1 nor x_2 is in (a, b), then $f(x_1) = f(x_2) = 0$. The definitions of x_1 and x_2 now imply $f(x) = 0$ for all $x \in [a, b]$. In this case, Theorem 2 follows by taking x_0 to be any element of (a, b). ∎

Questions: What is Rolle's Theorem saying geometrically (in terms of the graph of $y = f(x)$)? Why can the condition that f is continuous at every point in $[a, b]$ be replaced by a weaker condition? In the proof, did we need to use that $f(a) = f(b) = 0$ or would the condition $f(a) = f(b)$ have been sufficient?

Theorem 3 (The Mean Value Theorem). *Let a and b be real numbers with $a < b$, and let $f : [a, b] \mapsto \mathbb{R}$ with f continuous on $[a, b]$ and f differentiable on (a, b). Then there is an $x_0 \in (a, b)$ for which*

$$f(b) - f(a) = (b - a)f'(x_0).$$

Proof. Let

$$g(x) = f(x) - f(a) - \left(\frac{f(b) - f(a)}{b - a}\right)(x - a),$$

and apply Rolle's theorem. ∎

Questions: What is the Mean Value Theorem saying geometrically (in terms of the graph of $y = f(x)$)? Does Theorem 3 imply Theorem 2? Does Theorem 2 imply Theorem 3? (In other words, did you understand the proof of Theorem 3?) Under the conditions of Theorem 3, what conclusions can we make if $f'(x) = 0$ for all $x \in (a, b)$? What if we drop the conditions that f is continuous on $[a, b]$ and that $f(a)$ and $f(b)$ are defined?

Homework:

(1) Prove that for every real number k the polynomial $f(x) = x^5 - 5x + k$ has at most one root in $[-1, 1]$.

(2) Prove that if f and g are continuous on $[a, b]$ and differentiable on (a, b) and if $f'(x) = g'(x)$ for all $x \in (a, b)$, then there is a constant C such that $f(x) = g(x) + C$ for all $x \in [a, b]$.

(3) Suppose that $f : (a, b) \mapsto \mathbb{R}$ is such that $|f(x) - f(y)| \le M|x - y|^2$ for all x and y in (a, b). Prove that f is a constant on (a, b).

(4) Prove that $|\sin x - \sin y| \le |x - y|$ for all real numbers x and y. (You may use information about the derivative of $\sin x$.)

(5) Let a and b be real numbers with $a < b$, and let $f : [a, b] \mapsto \mathbb{R}$ with f continuous on $[a, b]$ and f differentiable on (a, b). Suppose that $f'(x) > 0$ for all $x \in (a, b)$. Prove that if $a \le x \le y \le b$, then $f(x) < f(y)$. (In other words, you are to prove that a continuous function on $[a, b]$ with a positive derivative on (a, b) is increasing on (a, b).)