MATH 555 NOTES
Derivatives

Definition. Let $f : E \mapsto \mathbb{R}$ with $E \subseteq \mathbb{R}$, and let $p \in E$ with p a limit point of E. If

$$\lim_{x \to p} \frac{f(x) - f(p)}{x - p}$$

exists, then f *is differentiable at* p (or f *has a derivative at* p). If f is differentiable at each point of E, then we say that f *is differentiable on* E.

Notation: We write $f'(p)$ for the derivative of f at p (if it exists).

Questions: What's another way of expressing the limit in the definition (think in terms of $f(p+h) - f(p)$ in the numerator)? Does $f'(x)$ mean the same thing as $f'(p)$? What other notation exists for $f'(x)$? Of what geometric significance is the quotient $(f(x) - f(p))/(x-p)$? Is $f(x) = x^5$ differentiable at 1? Is $f(x) = \sqrt{x}$ differentiable at 0? Is $f(x) = |x|$ differentiable at 0? How are the last three answers related to the graphs of $y = f(x)$?

Theorem 1. *Let $E \subseteq \mathbb{R}$, and let $f : E \mapsto \mathbb{R}$ be differentiable at p. Then f is continuous at p.*

Proof. Observe that

$$f(x) = \frac{f(x) - f(p)}{x - p}(x - p) + f(p).$$

The limit of the expressions $(f(x) - f(p))/(x-p)$, $x - p$, and $f(p)$ each exist as x approaches p, and it easily follows that $\lim_{x \to p} f(x) = f'(p)(p-p) + f(p) = f(p)$. ■

Theorem 2. *Let f and g be functions mapping $E \subseteq \mathbb{R}$ into $\mathbb{R}$ which are differentiable at p. Then*
(i) $f + g$ is differentiable at p and $(f+g)'(p) = f'(p) + g'(p)$.
(ii) fg is differentiable at p and $(fg)'(p) = f'(p)g(p) + f(p)g'(p)$.
(iii) f/g is differentiable at p and $(f/g)'(p) = (f'(p)g(p) - f(p)g'(p))/(g(p))^2$ provided $g(p) \neq 0$.

Proof. The proofs of (i) and (iii) are left to you (you may want to read the argument for (iii)). To prove (ii), rewrite $(fg)(x) - (fg)(p)$ as $g(x)(f(x) - f(p)) + f(p)(g(x) - g(p))$ and proceed. ■

Questions: How could you use Theorem 2 to prove that the derivative of x^n is nx^{n-1} if n is a positive integer? How could this be proved without Theorem 2? How could you show the same for n an arbitrary integer?

Theorem 3 (The Chain Rule). *Let $g : D \mapsto \mathbb{R}$ and $f : E \mapsto \mathbb{R}$ where $D \subseteq \mathbb{R}$, $E \subseteq \mathbb{R}$, and $g(D) \subseteq E$. If g is differentiable at p and f is differentiable at $g(p)$, then $f \circ g$ is differentiable at p and $(f \circ g)'(p) = f'(g(p))g'(p)$.*

Proof. Let $h = f \circ g$. Note that $g(x)$ is continuous at p by Theorem 1. Use that

$$g(x)-g(p) = (x-p)(g'(p)+u(x)) \quad \text{and} \quad f(g(x))-f(g(p)) = (g(x)-g(p))(f'(g(p))+v(x)),$$

where $u(x)$ and $v(x)$ approach 0 as x goes to p. Then

$$f(g(x)) - f(g(p)) = (g(x) - g(p))(f'(g(p)) + v(x)) = (x - p)(g'(p) + u(x))(f'(g(p)) + v(x)),$$

from which the result easily follows. ■

Homework:

(1) Using the definition above, prove that $f(x) = (x+1)/(2x-1)$ is differentiable at 1. (Do not use Theorem 2. You should, however, know how to use Theorem 2. You may use a result about limits that is similar to Theorem 2.)

(2) Prove part (i) of Theorem 2 above.

(3) (a) Prove that the derivative of $f(x) = \sqrt{x}$ exists at p for each $p > 0$. What is the value of $f'(p)$?

(b) Let $f(x) = \sqrt[4]{x + \sqrt{x + \sqrt{x}}}$. Using part (a) and the results above and indicating when and how you are using them, determine where f is differentiable and compute its derivative.

(4) (a) Let a and b be real numbers with $a < b$. Suppose that $f : (a, b) \mapsto \mathbb{R}$ is differentiable on (a, b). Prove or disprove that

$$\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

exists and equals $f'(x)$ for $x \in (a, b)$.

(b) Let a and b be real numbers with $a < b$. Suppose that

$$\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$$

exists for $x \in (a, b)$. Prove or disprove that $f'(x)$ exists on (a, b) and is equal to the above limit.