MATH 555 NOTES
More on Continuity

We mention one other result without giving a detailed proof. The fact that we are omitting the proof, however, is not an indication of the difficulty of the result. The result here is in fact easy to establish. You should read a proof in a text and see if you can follow the argument.

Theorem. *Let f be a function defined on $E \subseteq \mathbb{R}$, and let $p \in E$ be a limit point of E. Then the following are equivalent:*

(i) $f(x)$ is continuous at p.

(ii) $\lim_{x \to p} f(x)$ exists and equals $f(p)$.

(iii) For every sequence $\{x_j\}_{j=1}^{n}$ converging to p, the sequence $\{f(x_j)\}_{j=1}^{n}$ converges to $f(p)$.

Questions: If $\{a_n\}_{n=1}^{\infty}$ converges, what does this theorem tell us about the sequences $\{|a_n|\}_{n=1}^{\infty}$, $\{a_n^2\}_{n=1}^{\infty}$, and $\{\sqrt{|a_n|}\}_{n=1}^{\infty}$? Is the condition that $p \in E$ necessary in this theorem? What does this theorem imply about $f \circ g$ and continuity? If $f(p)$ is replaced by L in (ii) and (iii), are these two conditions still equivalent?

Homework:

(1) Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function, and let $y \in \mathbb{R}$. Prove that $\{x \in \mathbb{R} : f(x) \neq y\}$ is an open set.