MATH 555 NOTES
Uniform Continuity

Definition. Let $f : E \mapsto \mathbb{R}$ where $E \subseteq \mathbb{R}$. Then f *is uniformly continuous on* E (or f *is continuous uniformly on* E) if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all x and y in E satisfying $|x - y| < \delta$.

Comments: A comparison with the definition of continuity is useful. Observe that if f is uniformly continuous on E, then it is also continuous on E. It does not, however, make sense to refer to uniform continuity at a point. The "uniform" aspect of the terminology refers to the fact that not only is f continuous at each point $x_0 \in E$ but also that the δ that occurs in the definition of continuity at a point x_0 is independent of the x_0 one considers in E. In the definition of continuity at a point x_0, δ can depend on ε, x_0, and f; in the definition of uniform continuity, δ can depend on ε, f, and E but not on any particular point in E.

Questions and Problems: Is $f(x) = x^3$ uniformly continous on $(0, 1]$? What about $f(x) = 1/x$? For each of these $f(x)$, describe sets E such that $f(x)$ is uniformly continuous on E and describe sets E such that $f(x)$ is not uniformly continuous on E. Is $f(x) = |x|$ uniformly continuous on $(-1, 1)$ (what about on other sets)? Can a function f which is unbounded on a set E be uniformly continuous on E? How is uniform continuity of a function f related to slopes of lines passing through the graph of $y = f(x)$ at two points?

Theorem. *Let E be a compact set in $\mathbb{R}$. Suppose $f : E \mapsto \mathbb{R}$ is continuous. Then f is uniformly continuous on E.*

Proof. Let $\varepsilon > 0$. Since f is continuous on E, for each $x_0 \in E$, there is a $\delta(x_0) > 0$ such that if $|x - x_0| < \delta(x_0)$ and $x \in E$, then $|f(x) - f(x_0)| < \varepsilon/2$. Define $S(x_0) = (x_0 - \delta(x_0)/2, x_0 + \delta(x_0)/2)$. The sets $S(x_0)$ with $x_0 \in E$ form an open cover of E. Since E is compact, there are $x_1, \dots, x_n$ in E such that $E \subseteq \bigcup_{j=1}^{n} S(x_j)$. Let $\delta = \min_{1 \le j \le n} \{\delta(x_j)/2\}$. We complete the proof by showing that if x and y are in E and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Let x and y be in E with $|x - y| < \delta$. Since $x \in E$ and $E \subseteq \bigcup_{j=1}^{n} S(x_j)$, we deduce that $x \in S(x_k)$ for some $k \in \{1, 2, \dots, n\}$. Observe that $|x - x_k| < \delta(x_k)/2 < \delta(x_k)$. Since $x \in E$, we obtain $|f(x) - f(x_k)| < \varepsilon/2$. The definition of δ and the triangle inequality imply

$$|y - x_k| \le |y - x| + |x - x_k| < \delta + \frac{\delta(x_k)}{2} \le \delta(x_k).$$

Since $y \in E$, we obtain $|f(y) - f(x_k)| < \varepsilon/2$. From the triangle inequality, we now obtain

$$|f(x) - f(y)| \le |f(x) - f(x_k)| + |f(y) - f(x_k)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

completing the proof. ■

Questions: What examples can you come up with now of functions f with domain E such that f is uniformly continuous on E? What theorem from before about compact sets helps come up with examples? Are there examples where f is uniformly continuous on a set E but E is not compact? (In other words, were you awake earlier during this lecture?)

Homework:

(1) Prove that $\sqrt{x}$ is uniformly continuous on $(1, \infty)$.

(2) Let f and g be uniformly continuous real-valued functions defined on a set $E \subseteq \mathbb{R}$.

(a) Prove that $f + g$ is uniformly continuous on E.

(b) Explain why it is not necessarily the case that fg is uniformly continuous on E.

(c) Show by example that even if in (b) one of the functions f or g is bounded, the product fg still may not be uniformly continuous.

(d) Prove that if f and g are both bounded, then the product fg is uniformly continuous.

(3) (a) Suppose that f and g are continuous real-valued functions defined on a set $E \subseteq \mathbb{R}$ such that $f + g$ is uniformly continuous on E. Must it be the case that f and g are each uniformly continuous on E? Must it be the case that at least one of f or g is uniformly continuous on E? (Be sure to address both questions.)

(b) Suppose that f and g are continuous real-valued functions defined on a set $E \subseteq \mathbb{R}$ such that fg is uniformly continuous on E. Must it be the case that f and g are each uniformly continuous on E? Must it be the case that at least one of f or g is uniformly continuous on E? (Be sure to address both questions.)

(4) Let $E \subseteq \mathbb{R}$ be bounded, and let f be a uniformly continuous real-valued function on E. Prove that f is bounded on E (i.e., prove that the set $\{f(x) : x \in E\}$ is bounded).

(5) Let A and B be two disjoint sets of real numbers. Let f be a real-valued function which is uniformly continuous on A and on B. Show by example that f may not be uniformly continuous on $A \cup B$.