MATH 555 NOTES
A Necessary and Sufficient Condition for Riemann Integrability

The main purpose of this section is to establish Lebesgue's necessary and sufficient condition for a bounded function on $[a, b]$ to be Riemann integrable. We begin with some preliminaries.

Lemma 1. *If $S_1, S_2, \ldots$ are sets of real numbers each with measure zero, then their union has measure zero.*

Proof. Let $\varepsilon > 0$. For each positive integer j, the set S_j can be covered by a countable set T_j of open intervals with total length $\varepsilon/2^j$. In other words,

$$S_j \subseteq \bigcup_{A \in T_j} A \qquad \text{and} \qquad \sum_{A \in T_j} |A| < \frac{\varepsilon}{2^j}.$$

Taking $T = \bigcup_{j=1}^{\infty} T_j$, we see that T is a countable set of open intervals covering $\bigcup_{j=1}^{\infty} S_j$ and $\sum_{A \in T} |A| < \varepsilon$. This completes the proof. ■

Definition. Let f be a bounded real-valued function on $[a, b]$. Let $S \subseteq [a, b]$. Then the *oscillation of f on S* is

$$\Omega_f(S) = \sup\{f(x) - f(y) : x \in S, y \in S\}.$$

The *oscillation of f at a point* $x_0 \in [a, b]$ is $\omega_f(x_0) = \lim_{\varepsilon \to 0+} \Omega_f(\{x \in [a, b] : |x - x_0| < \varepsilon\})$.

Question: Why does the limit in this definition exist?

Lemma 2. *Let f be a bounded real-valued function on $[a, b]$, and let $\varepsilon > 0$. Suppose that $\omega_f(x_0) < \varepsilon$ for every $x_0 \in [a, b]$. There exists a $\delta = \delta(\varepsilon) > 0$ such that if I is a closed interval in $[a, b]$ with length $|I| < \delta$, then $\Omega_f(I) < \varepsilon$.*

Proof. For each $x_0 \in [a, b]$, the inequality $\omega_f(x_0) < \varepsilon$ implies that there is a $\delta(x_0) > 0$ for which the oscillation of f on the set $S(x_0) = \{x \in [a, b] : |x - x_0| < \delta(x_0)\}$ is $< \varepsilon$. The sets $T(x_0) = \{x \in \mathbb{R} : |x - x_0| < \delta(x_0)/2\}$ form an open covering of $[a, b]$ and $[a, b]$ is compact. Hence, this open covering of $[a, b]$ contains a finite subcover $\{T(x_1), \ldots, T(x_n)\}$. Let $\delta = \min\{\delta(x_j)/2 : 1 \le j \le n\}$. Let I be a closed interval in $[a, b]$ with $|I| < \delta$. Consider arbitrary x and y in I. Since $x \in [a, b]$, there is a $j \in \{1, 2, \ldots, n\}$ such that $x \in T(x_j)$. Observe that $x \in S(x_j)$ and $|T(x_j)| < \delta$. Now, $|I| < \delta$ and $x \in T(x_j)$ implies

$$|y - x_j| \le |y - x| + |x - x_j| < \delta + \delta \le \delta(x_j).$$

Hence, $y \in S(x_j)$. Since both x and y are in $S(x_j)$ and the oscillation of f on $S(x_j)$ is $< \varepsilon$, we obtain $|f(x) - f(y)| < \varepsilon'$ where $\varepsilon' = \max_{1 \le j \le n} \{\Omega_f(S(x_j))\} < \varepsilon$. It follows that the oscillation of f on I is $\le \varepsilon' < \varepsilon$. ■

Lemma 3. *Let f be a bounded real-valued function on $[a, b]$, and let $\varepsilon > 0$. Then the set $W(\varepsilon) = \{x \in [a, b] : \omega_f(x) \geq \varepsilon\}$ is closed.*

Proof. Let $x_0 \in \mathbb{R} - W(\varepsilon)$. If $x_0 \in (-\infty, a) \cup (b, \infty)$, then clearly for some $\varepsilon' > 0$, the interval $(x_0 - \varepsilon', x_0 + \varepsilon')$ is in $\mathbb{R} - W(\varepsilon)$. We show the same is true if $x_0 \in [a, b]$ (as well as $\mathbb{R} - W(\varepsilon)$). Note that $\omega_f(x_0) < \varepsilon$. Let t be in the open interval $(\omega_f(x_0), \varepsilon)$. By the definition of $\omega_f(x_0)$, there is an $\varepsilon' > 0$ such that if $|x - x_0| < \varepsilon'$ and $|y - x_0| < \varepsilon'$ and x and y are in $[a, b]$, then $|f(x) - f(y)| < t$. Suppose $x' \in (x_0 - \varepsilon', x_0 + \varepsilon')$, and let $\varepsilon'' = \varepsilon' - |x' - x_0|$. Note that $\varepsilon'' > 0$. Observe that if $|x - x'| < \varepsilon''$ and $|y - x'| < \varepsilon''$ and x and y are in $[a, b]$, then $|x - x_0| < \varepsilon'$ and $|y - x_0| < \varepsilon'$ and, hence, $|f(x) - f(y)| < t$. Therefore, either $x' \notin [a, b]$ or $\omega_f(x') \leq t < \varepsilon$. In either case, we deduce that $x' \notin W(\varepsilon)$. Thus, $(x_0 - \varepsilon', x_0 + \varepsilon')$ is in $\mathbb{R} - W(\varepsilon)$. It follows that $\mathbb{R} - W(\varepsilon)$ is open and, hence, $W(\varepsilon)$ is closed. ■

Theorem. *Let a and b be real numbers with $a < b$. Let f be a bounded real-valued function on $[a, b]$. Then f is Riemann integrable on $[a, b]$ if and only if the set of discontinuities of f on $[a, b]$ has measure zero.*

Proof. Let $S \subseteq [a, b]$ be the set of discontinuities of f. First, we consider the case that S does not have measure zero and prove that f is not Riemann integrable on $[a, b]$. Assume that f is Riemann integrable on $[a, b]$. Let $W(\varepsilon)$ be as defined in Lemma 3, and observe that by Homework (1) below, $x \in S$ if and only if $x \in W(1/n)$ for some positive integer n. In other words, $S = \bigcup_{n=1}^{\infty} W(1/n)$. By Lemma 1, since S does not have measure zero, there is a positive integer n for which $W(1/n)$ does not have measure zero. It follows that there is a $t > 0$ such that $W(1/n)$ cannot be covered by a countable collection of open intervals of total length $< t$. Fix such a t. Since f is Riemann integrable on $[a, b]$, there is a partition $\mathcal{P} = \{x_0, x_1, \ldots, x_m\}$ of $[a, b]$ for which

$$U(\mathcal{P}, f) - L(\mathcal{P}, f) < \frac{t}{2n}.$$

Consider the open intervals $A_j = (x_j - (t/2^{j+3}), x_j + (t/2^{j+3}))$ for $j \in \{0, 1, \ldots, m\}$ together with the open intervals $B_j = (x_{j-1}, x_j)$ for $j \in \{1, \ldots, m\}$. Note that every element of $[a, b]$ and, hence, of $W(1/n)$ must belong to one of these intervals. Also, $\sum_{j=0}^{m} |A_j| < t/2$. Let J be the set of $j \in \{1, \ldots, m\}$ for which $B_j \cap W(1/n)$ is non-empty. By the definition of t, $\sum_{j \in J} |B_j| > t/2$. Observe that if $j \in J$, then there is an $x' \in B_j \cap W(1/n)$ so that $M_j(f) - m_j(f) \geq \omega_f(x') \geq 1/n$. Therefore,

$$\begin{aligned} U(\mathcal{P}, f) - L(\mathcal{P}, f) &= \sum_{j=1}^{m} (M_j(f) - m_j(f))(x_j - x_{j-1}) \geq \sum_{j \in J} (M_j(f) - m_j(f))(x_j - x_{j-1}) \\ &\geq \frac{1}{n} \sum_{j \in J} (x_j - x_{j-1}) = \frac{1}{n} \sum_{j \in J} |B_j| > \frac{t}{2n}, \end{aligned}$$

a contradiction to the definition of $\mathcal{P}$. Hence, f is not Riemann integrable on $[a, b]$.

Now, we consider the case that S has measure zero and prove that f must be Riemann integrable on $[a, b]$. Let $\varepsilon > 0$. We show that there is a partition $\mathcal{P}$ of $[a, b]$ for which $U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon$. As before, we have $S = \bigcup_{n=1}^{\infty} W(1/n)$. Since $W(1/n) \subseteq S$ for each positive integer n, we easily deduce that each $W(1/n)$ has measure zero. Let $M = \sup\{f(x) : x \in [a, b]\}$ and $m = \inf\{f(x) : x \in [a, b]\}$. We fix $n > (M - m + b - a)/\varepsilon$. Since $W(1/n)$ has measure zero, there is a countable collection of open intervals $A_1, A_2, \ldots$ that cover $W(1/n)$ and for which $\sum_{j=1}^{\infty} |A_j| < 1/n$. By Lemma 3, the set $W(1/n)$ is closed. Since $W(1/n) \subseteq [a, b]$, $W(1/n)$ is compact. Hence, there is a finite set J of positive integers for which $\{A_j : j \in J\}$ is a covering of $W(1/n)$. By altering these A_j's if necessary (i.e., if two or more overlap, combine them to make one open interval, decreasing the total of the lengths), we can find a finite set of open intervals $B_0 = (b_0, b_1), \ldots, B_r = (b_{2r}, b_{2r+1})$ satifying

$$b_0 \le b_1 \le \cdots \le b_{2r+1}, \quad \sum_{j=0}^{r} (b_{2j+1} - b_{2j}) < \frac{1}{n}, \quad \text{and} \quad W(1/n) \subseteq \bigcup_{j=0}^{r} B_j.$$

For each $j \in \{1, \ldots, r\}$, the elements x of $[b_{2j-1}, b_{2j}]$ satisfy $\omega_f(x) < 1/n$. Thus, if $b_{2j-1} \ne b_{2j}$, then by Lemma 2 there is a $\delta(j)$ for which every closed interval $I \subseteq [b_{2j-1}, b_{2j}]$ with $|I| < \delta(j)$ satisfies $\Omega_f(I) < 1/n$. There is similarly a δ' and a δ'' such that if $a < b_0$ or $b_{2r+1} < b$, respectively, then every closed interval I in $[a, b_0]$ with $|I| < \delta'$ and every closed interval I in $[b_{2r+1}, b]$ with $|I| < \delta''$ satisfies $\Omega_f(I) < 1/n$. We define $\mathcal{P} = \{x_0, x_1, \ldots, x_s\}$ to be a partition of $[a, b]$ which includes the elements of $\{a, b, b_0, b_1, \ldots, b_{2r+1}\} \cap [a, b]$ and which satisfies $\|\mathcal{P}\| < \delta = \min\{\delta', \delta'', \delta(1), \ldots, \delta(r)\}$. Let T_1 denote the set of $j \in \{1, \ldots, s\}$ for which $[x_{j-1}, x_j]$ contains no element from the union of $B_0, B_1, \ldots, B_r$, and let T_2 denote the remaining $j \in \{1, \ldots, s\}$. Observe that $M_j(f) - m_j(f) \le M - m$ for all j so that

$$\sum_{j \in T_2} (M_j(f) - m_j(f))(x_j - x_{j-1}) \le (M - m) \sum_{j \in T_2} (x_j - x_{j-1}) \le (M - m) \sum_{j=0}^{r} |B_j| < \frac{M - m}{n}.$$

On the other hand, for each $j \in T_1$, the interval $I = [x_{j-1}, x_j]$ having length $< \delta$ implies that $\Omega_f(I) < 1/n$ so that $M_j(f) - m_j(f) < 1/n$. Hence,

$$\sum_{j \in T_1} (M_j(f) - m_j(f))(x_j - x_{j-1}) \le \frac{1}{n} \sum_{j \in T_1} (x_j - x_{j-1}) \le \frac{1}{n} \sum_{j=1}^{s} (x_j - x_{j-1}) = \frac{b - a}{n}.$$

It follows that

$$U(\mathcal{P}, f) - L(\mathcal{P}, f) = \sum_{j=1}^{s} (M_j(f) - m_j(f))(x_j - x_{j-1})$$

$$= \sum_{j \in T_1} (M_j(f) - m_j(f))(x_j - x_{j-1}) + \sum_{j \in T_2} (M_j(f) - m_j(f))(x_j - x_{j-1})$$

$$< \frac{M - m + b - a}{n} < \varepsilon.$$

Thus, f is Riemann integrable on $[a, b]$, completing the proof. ■

Homework:

(1) Let f be a real-valued function on $[a, b]$, and let $x_0 \in [a, b]$. Prove that f is continuous at x_0 if and only if $\omega_f(x_0) = 0$.

(2) Let $g \in R[a, b]$. Let M and m be such that $m \leq g(x) \leq M$ for all $x \in [a, b]$. Suppose f is continuous on $[m, M]$. Using the theorem from this section, explain why $f \circ g \in R[a, b]$.

(3) Define f on $[0, 1]$ as follows. If x is irrational, set $f(x) = 0$. If $x = a/b$ is rational with a and b relatively prime non-negative integers and $b > 0$, then set $f(x) = 1/b$. In particular, $0 = 0/1$ and $1 = 1/1$ implies $f(0) = f(1) = 1$. Determine with proof whether f is Riemann integrable on $[0, 1]$.