MATH 555 NOTES
Sets of Measure Zero

Definition. Let $S \subseteq \mathbb{R}$. We say that the set S *has measure* 0 if for every $\varepsilon > 0$, there is a sequence $\{A_n\}_{n=1}^{\infty}$ of open intervals, say $A_n = (a_n, b_n)$ with $a_n < b_n$, such that

$$S \subseteq \bigcup_{n=1}^{\infty} A_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} (b_n - a_n) < \varepsilon.$$

In other words, S has measure 0 if for every $\varepsilon > 0$, S can be covered by a countable number of open intervals with the sum of the lengths of the open intervals being $< \varepsilon$.

Questions: Does a finite set S have measure 0? Can an infinite set S have measure 0? Does every countable set have measure 0? Is it possible for the set of discontinuities of a function not to have measure 0?

Example: The main purpose of this section is to illustrate the above definition by showing that there exists an uncountable set with measure 0. We use the classical approach and discuss the *Cantor set*. The Cantor set can be defined as follows. Beginning with $T_0 = [0, 1]$ (a single closed interval), we remove the middle third $(1/3, 2/3)$ of T_0 to form a new set T_1. Thus, $T_1 = [0, 1] - (1/3, 2/3) = [0, 1/3] \cup [2/3, 1]$, a union of two closed intervals. Next, we remove from each of these closed intervals their middle thirds to form T_2 so that $T_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$, a union of four closed intervals. We continue so that for each $n \geq 0$, T_n is the union of exactly 2^n closed intervals each of length $1/3^n$, and to form T_{n+1} we remove the middle thirds of each of the 2^n intervals which form T_n. We take $C = \bigcap_{n=1}^{\infty} T_n$. The set C is called the Cantor set.

Theorem. *The Cantor set described above is an uncoutable set with measure* 0.

Proof. We begin by showing that C has measure 0. Let $\varepsilon > 0$. Consider n such that $(2/3)^n < \varepsilon/2$. It follows that the 2^n closed intervals, each of length $1/3^n$, forming T_n can be covered by a union of 2^n open intervals having total length $< \varepsilon/2$. Since $C \subseteq T_n$, these open intervals also cover C. Taking the A_n's in the definition above to be these open intervals together with a countable number of any other open intervals with total length $< \varepsilon/2$, we deduce that C has measure 0.

Next, we prove that C is uncountable. For this purpose, we consider the ternary (base 3) representation of the elements of C. Before doing so, we clarify what we mean by the ternary representation of a real number. For every real number $\alpha \in [0, 1]$, there exist unique numbers $d_j \in \{0, 1, 2\}$ with $j \in \mathbb{Z}^+$ satisfying:

(i) if there is a maximal $k \in \mathbb{Z}^+$ for which $d_k \neq 0$, then $d_k = 2$, and

(ii) $\alpha = \displaystyle\sum_{j=1}^{\infty} d_j 3^{-j}$.

The representation in (ii) we call the ternary representation of α; that this representation does in fact exist and is unique should be justified (see Homework Problem (1)). We refer to the d_j as the ternary digits (even though there are not 10 of them). Then it can be shown by induction that T_n is exactly the set of $\alpha \in [0,1]$ with each of $d_1, d_2, \dots, d_n$ in $\{0,2\}$. It follows that C is the set of all numbers in $[0,1]$ having each of its ternary digits being either 0 or 2. By considering a natural one-to-one mapping of C onto $[0,1]$ or by doing a Cantor diagonal argument, we obtain that C is uncountable. ■

Questions: In the proof above, what is the purpose of condition (i) for the ternary representation of α? How might the above description of ternary representations be contrary to what you would like to call ternary representations? Does this make the above proof wrong? Is C open or closed or neither or both? Is C compact?

Homework:

(1) Let $\alpha \in [0,1]$. Prove the ternary representation of α described in the proof above exists and is unique.

(2) Suppose we remove closed intervals instead of open intervals when defining C. Thus, $T_1 = [0,1] - [1/3, 2/3] = [0,1/3) \cup (2/3, 1]$ and $T_2 = [0,1/9) \cup (2/9,1/3) \cup (2/3,7/9) \cup (8/9,1]$. Defining $C = \bigcap_{n=1}^{\infty} T_n$ as before (but with the revised T_n), determine whether C is an uncountable set with measure 0. Justify your answer.