MATH 555 NOTES
The Intermediate Value Theorem

Something to Ponder: A few years ago, when driving down Elmwood, I noticed the speed limit was 35 miles per hour. As I approached the interstate on Elmwood, I saw a second sign posted that read, "Minimum Speed 45 mph". Presumably this sign was indicating a speed limit change. Prior to this second sign, I was allowed to drive a maximum of 35 miles per hour, and after the sign I was allowed to drive 45 or more miles per hour, but not less than that. What's not right here? (Note that the signs have since changed.)

Theorem (Intermediate Value Theorem). *Let $a < b$ and let $f : [a, b] \mapsto \mathbb{R}$ be continuous. Suppose that t is such that $f(a) < t < f(b)$ (or $f(b) < t < f(a)$). Then there is a $c \in (a, b)$ for which $f(c) = t$.*

Definition. Let S be a nonempty set of real numbers that is bounded above. Then a is the *least upper bound* of S (denoted $a = \text{l.u.b.}(S)$) or a is the *supremum* of S (denoted $a = \sup(S)$) if a is an upper bound of S and if whenever b is an upper bound of S we have $b \geq a$.

Comments: The notations are interchangable. Also, there is an analogous definition (and notation) for the *greatest lower bound* of a set S or the *infimum* of a set S.

Questions: What are $\sup(S)$ and $\inf(S)$ for each of the following: $S = (0, 1)$, $S = (0, 1) \cap \mathbb{Q}$, and $S = \{n/(n + 1) : n \in \mathbb{Z}^+\}$?

Least Upper Bound Property. *Every nonempty set of real numbers that is bounded from above has a least upper bound.*

Comment: The least upper bound property we will view as an axiom about the real numbers and use it to prove the Intermediate Value Theorem. It has been used to prove several results that you have already seen, for example that bounded monotone sequences converge. On the other hand, this is not strictly speaking necessary. Alternatively, we can assume (as an axiom) that every bounded monotone sequence converges and establish results such as the Intermediate Value Theorem (and even the least upper bound property).

Proof of the Intermediate Value Theorem. Let $S = \{x \in [a, b] : f(x) < t\}$. Observe that $a \in S$ so that $S \neq \emptyset$. Set $c = \sup(S)$. Since b is an upper bound for S, $c \leq b$. Also, since $a \in S$, $c \geq a$. We claim that $c \in (a, b)$ and $f(c) = t$. Assume $f(c) \neq t$, and let $t' = f(c)$. We consider two cases: (i) $t > t'$ and (ii) $t < t'$. Note that in case (i), we have $c < b$ since $c \leq b$ and $f(c) = t' < t < f(b)$. Similarly, in case (ii), we have $c > a$ since $c \geq a$ and $f(c) = t' > t > f(a)$. In either case, let $\varepsilon = |t - t'| > 0$. Since f is continuous at c, there is a $\delta > 0$ such that if $|x - c| < \delta$ and $x \in [a, b]$, then $|f(x) - f(c)| < \varepsilon$. In the case of (i), we deduce that $f(x) < f(c) + \varepsilon = f(c) + t - t' = t$ so that each $x \in (c, c + \delta) \cap [a, b]$ (which is non-empty since $a \leq c < b$) is such that $x \in S$, contradicting that c is an upper bound of S. In the case of (ii), we deduce that $f(c) - f(x) < \varepsilon = t' - t$ so that for $x \in (c - \delta, c) \cap [a, b]$ (which is non-empty since $a < c \leq b$), we have $f(x) > t$. In fact, $t > f(a)$ implies now that $a \notin (c - \delta, c)$, so we can deduce that $c - \delta$ is an upper bound of S, contradicting that c

is the least upper bound of S. Hence, in either case, we get a contradiction. This implies $f(c) = t$. The inequalities $f(a) < t < f(b)$ and $a \leq c \leq b$ now imply $a < c < b$, completing the proof. ∎

Homework:

(1) Let S be a non-empty set of real numbers bounded above by a. Prove that a is the least upper bound of S if and only if for each $\varepsilon > 0$ there is at least one element $s \in S$ such that $a - \varepsilon < s \leq a$.

(2) Prove that if $f(x)$ is a polynomial with real coefficients of odd degree, then $f(x)$ has a real root.

(3) Let $f : [0, 1] \mapsto [0, 1]$ be a continuous function. Prove that there is an $x \in [0, 1]$ such that $f(x) = x$.

(4) Prove that for some $x \in (0, \pi/2)$, $\cos x = x$.

(5) Suppose that $f : [-1, 1] \mapsto \mathbb{R}$ is continuous and that $f(-1) = f(1)$. Prove that for some $t \in [0, 1]$, $f(t) = f(t - 1)$.