MATH 555 NOTES
Two More Theorems on Sequences of Functions

In this section, we illustrate two further consequences of sequences of functions which converge uniformly. Questions to ask yourself as you read the theorems are, "What happens if the sequence of functions converges pointwise? What are good examples to illustrate any difference this might make?"

Theorem. *Let $\{f_n\}$ be a sequence of functions converging uniformly to f on $E \subseteq \mathbb{R}$. Suppose for each positive integer n, the function f_n is bounded on E. Then f is bounded on E.*

Proof. Since f_n converges uniformly to f on E, there is an N such that if $n \geq N$, then $|f_n(x) - f(x)| < 1$ for all $x \in E$. Fix $n \geq N$. Then $|f(x)| < |f_n(x)| + 1$ for all $x \in E$. Also, since f_n is bounded on E, we have that there is an M such that $|f_n(x)| \leq M$ for all $x \in E$. It follows that for each $x \in E$, $|f(x)| < M + 1$. Thus, f is bounded on E. ■

Theorem. *Let a and b be real numbers with $a < b$. Let $\{f_n\}$ be a sequence of functions converging uniformly to a function f on $[a, b]$. Suppose for each positive integer n, we have $f_n \in \mathcal{R}[a, b]$. Then $f \in \mathcal{R}[a, b]$. Furthermore, if $F_n(t) = \int_a^t f_n$ and $F(t) = \int_a^t f$ for $t \in [a, b]$, then $\{F_n\}$ converges uniformly to F on $[a, b]$.*

Proof. Let $\varepsilon > 0$. Since f_n converges uniformly to f on E, there is an N such that if $n \geq N$, then $|f_n(x) - f(x)| < \varepsilon/(3(b-a))$ for all $x \in E$. Fix such an N for the rest of the proof. Fix $r \geq N$. Consider and arbitrary partition $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$. Observe that if L is a lower bound on f_r in $[x_{j-1}, x_j]$, then

$$f(x) \geq f_r(x) - \frac{\varepsilon}{3(b-a)} \geq L - \frac{\varepsilon}{3(b-a)} \qquad \text{for all } x \in [a, b].$$

It follows that $m_j(f) \geq m_j(f_r) - (\varepsilon/(3(b-a)))$. Similarly $m_j(f_r) \geq m_j(f) - (\varepsilon/(3(b-a)))$. Hence,

$$|L(\mathcal{P}, f) - L(\mathcal{P}, f_r)| \leq \sum_{j=1}^{n} |m_j(f) - m_j(f_r)|(x_j - x_{j-1}) \leq \frac{\varepsilon}{3(b-a)} \sum_{j=1}^{n} (x_j - x_{j-1}) = \frac{\varepsilon}{3}.$$

Similarly,

$$|U(\mathcal{P}, f) - U(\mathcal{P}, f_r)| \leq \frac{\varepsilon}{3}.$$

We deduce that

$$\begin{aligned} U(\mathcal{P}, f) - L(\mathcal{P}, f) &= |U(\mathcal{P}, f) - L(\mathcal{P}, f)| \\ &\leq |U(\mathcal{P}, f) - U(\mathcal{P}, f_r)| + |U(\mathcal{P}, f_r) - L(\mathcal{P}, f_r)| + |L(\mathcal{P}, f_r) - L(\mathcal{P}, f)| \\ &\leq |U(\mathcal{P}, f_r) - L(\mathcal{P}, f_r)| + \frac{2\varepsilon}{3}. \end{aligned}$$

Since f_r is Riemann integrable, we can find a partition $\mathcal{P}$ for which the right-hand side above is $< \varepsilon$. The inequality above then implies that f is Riemann integrable. Also, for $n \geq N$, we have for every $t \in [a, b]$ that

$$\begin{aligned} |F_n(t) - F(t)| &= \left| \int_a^t f_n - \int_a^t f \right| = \left| \int_a^t (f_n - f) \right| \\ &\leq \int_a^t |f_n - f| \leq \frac{\varepsilon}{3(b-a)} \int_a^t 1 = \frac{\varepsilon(t-a)}{3(b-a)} < \varepsilon. \end{aligned}$$

Thus, $\{F_n\}$ converges uniformly to F on $[a, b]$. ■

Additional Question: Why do $F_n(t)$ and $F(t)$ exist in the statement of the last theorem?

Comment: To fully answer the questions at the beginning of the section, you should consider if f_n converges to f pointwise in the last theorem, both the possibility that f is not Riemann integrable and the possibility that f is but F_n does not converge uniformly (or pointwise) to F. Consider examples discussed in the previous notes in discussing questions about the above two theorems. It might help also to come up with an example of a sequence f_n with a finite number of discontinuities on $E = [0, 1]$ for each n but such that the limiting function f is discontinuous everywhere on $[0, 1]$.

Homework:

(1) Let $f_n(x) = \sum_{j=0}^{n} x^j$ and $f(x) = 1/(1-x)$. Prove that $\{f_n\}$ converges uniformly to f on $[a, b]$ for any fixed real numbers a and b satisfying $-1 < a < b < 1$.

(2) Let a and b be as in Problem (1) above. Using Problem (1) and the second theorem in this section, prove that $\left\{ \sum_{j=0}^{n} \frac{x^j}{j+1} \right\}$ converges uniformly to $-\log(1-x)$ on $[a, b]$. Also, explain why this implies that the $\left\{ \sum_{j=0}^{n} \frac{x^j}{j+1} \right\}$ converges pointwise to $-\log(1-x)$ on $(-1, 1)$. (In other words, explain why the series $\sum_{j=0}^{\infty} \frac{x^j}{j+1}$ converges to $-\log(1-x)$ on $(-1, 1)$.)