MATH 555 NOTES
Sequences of Functions - Pointwise and Uniform Convergence

We briefly discuss the topic of sequences of real valued functions. The notion is clear; we want a sequence $\{f_n\} = \{f_n\}_{n=1}^{\infty}$ such that each function f_n maps a subset of $\mathbb{R}$ into $\mathbb{R}$. We will be interested in the convergence of such a sequence. In particular, there will be two notions of convergence we will want to make use of.

Definition 1. Let $\{f_n\}$ be a sequence of real valued functions, and let $E \subseteq \mathbb{R}$ be contained in the domain of each f_n. The sequence of functions $\{f_n\}$ *converges pointwise on* E provided that for each $x \in E$, we have $\lim\limits_{n\to\infty} f_n(x)$ exists. If we denote this limit by $f(x)$, then f is called the *limit of the sequence* $\{f_n\}$ and we say that $\{f_n\}$ *converges pointwise to* f *on* E.

Definition 2. Let $\{f_n\}$ be a sequence of real valued functions, and let $E \subseteq \mathbb{R}$ be contained in the domain of each f_n. The sequence of functions $\{f_n\}$ *converges uniformly on* E if there is a function $f : E \mapsto \mathbb{R}$ such that for every $\varepsilon > 0$, there exists an N for which $|f_n(x) - f(x)| < \varepsilon$ for every $n \geq N$ and for every $x \in E$. In this case, we say that $\{f_n\}$ *converges uniformly to* f *on* E.

Questions: If $f_n(x) = x^n$ and $E = [0, 1]$, what is the limit $f(x)$ of the sequence $\{f_n\}$? Does $\{f_n\}$ converge uniformly to $f(x)$ on E (consider x near 1)? What if $E = [0, 1/2]$? What happens if $f_n(x) = x^{1/n}$ and $E = [0, 1]$? What happens if $f_n(x) = n^2$ for $0 < x < 1/n$ and $f_n(x) = 0$ for $x = 0$ and for $1/n \leq x \leq 10$ and if $E = [0, 10]$? Does uniform convergence of $\{f_n\}$ imply pointwise convergence?

Theorem. *If $\{f_n\}$ is a sequence of real valued continuous functions on E which converges uniformly to f on E, then f is continuous on E.*

Proof. Let $x_0 \in E$. Let $\varepsilon > 0$. To establish the theorem, we show there exists a $\delta > 0$ such that if $0 < |x - x_0| < \delta$ and $x \in E$, then $|f(x) - f(x_0)| < \varepsilon$. Let N be such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3} \qquad \text{for all } n \geq N \text{ and for all } x \in E.$$

Fix $n \geq N$. Since f_n is continuous at x_0, there is a $\delta > 0$ such that if $0 < |x - x_0| < \delta$ and $x \in E$, then

$$|f_n(x) - f_n(x_0)| < \frac{\varepsilon}{3}.$$

Thus, if $0 < |x - x_0| < \delta$ and $x \in E$, then

$$|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

The theorem follows. ■

Questions: Can this theorem help answer some of the previous questions we considered above? Does the same result hold if instead of each f_n being continuous on E, we only have that for each $x_0 \in E$ there are infinitely many n for which f_n is continous at x_0?

Homework:

(1) For each positive integer n, let $f_n(x) = (x^n + 1)/(x^n + 2)$. Using the theorem in this section, prove that $\{f_n\}$ does not converge uniformly on $[0, 1]$.

(2) Give an example of a sequence of functions $\{f_n\}$ with each f_n not continous at each point in $[0, 1]$ and such that $\{f_n\}$ converges uniformly to some function f on $[0, 1]$ with f continous on $[0, 1]$.

(3) Suppose $f : \mathbb{R} \mapsto \mathbb{R}$ is uniformly continuous. For each positive integer n, define f_n by

$$f_n(x) = f\left(x + \frac{1}{n}\right) \quad \text{for } x \in \mathbb{R}.$$

Prove that $\{f_n\}$ converges uniformly to f on $\mathbb{R}$.