MATH 555 NOTES
A Connection Between Riemann and Riemann-Stieltjes Integrals

In the previous section, we dealt with some results related to evaluating Riemann-Stieltjes integrals. There are a couple more useful formulas to know which we discuss in this and the next section. The purpose of this section is to point out that sometimes (and not just when $g(x) = x$) we can interpret a Riemann-Stieltjes integral as a Riemann integral.

Theorem. *Suppose $f \in \mathcal{R}[a, b]$ and g is an increasing real valued differentiable function on $[a, b]$ for which $g' \in \mathcal{R}[a, b]$. Then $fg' \in \mathcal{R}[a, b]$, $f \in \mathcal{R}_g[a, b]$, and*

$$\int_a^b f\,dg = \int_a^b fg'.$$

Proof. Since $f \in \mathcal{R}[a, b]$ and $g' \in \mathcal{R}[a, b]$, we deduce immediately that $fg' \in \mathcal{R}[a, b]$. Let $\varepsilon > 0$. Since $g' \in \mathcal{R}[a, b]$, g' is bounded on $[a, b]$. Let N be a bound for g' on $[a, b]$. Since g is increasing we have $0 \le g'(t) \le N$ for all $t \in [a, b]$. Let $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$ for which $U(\mathcal{P}, f) - L(\mathcal{P}, f) < \varepsilon/N$ (which is possible since $f \in \mathcal{R}[a, b]$). By the Mean Value Theorem, for each $j \in \{1, 2, \dots, n\}$, there exists $t_j \in (x_{j-1}, x_j)$ such that

$$g(x_j) - g(x_{j-1}) = g'(t_j)(x_j - x_{j-1}).$$

We deduce that

$$\begin{aligned}
U(\mathcal{P}, f, g) - L(\mathcal{P}, f, g) &= \sum_{j=1}^n (M_j(f) - m_j(f))(g(x_j) - g(x_{j-1})) \\
&= \sum_{j=1}^n (M_j(f) - m_j(f))g'(t_j)(x_j - x_{j-1}) \\
&\le N \sum_{j=1}^n (M_j(f) - m_j(f))(x_j - x_{j-1}) = N\big(U(\mathcal{P}, f) - L(\mathcal{P}, f)\big) < \varepsilon.
\end{aligned}$$

It follows that $f \in \mathcal{R}_g[a, b]$.

Now, we consider any partitions $\mathcal{P}$ and $\mathcal{Q}$ of $[a, b]$ and consider the partition $\mathcal{P} \cup \mathcal{Q} = \{x_0, x_1, \dots, x_n\}$. For $j \in \{1, 2, \dots, n\}$, define t_j as before. Then

$$\begin{aligned}
L(\mathcal{P}, fg') \le L(\mathcal{P} \cup \mathcal{Q}, fg') &= \sum_{j=1}^n m_j(fg')(x_j - x_{j-1}) \\
&\le \sum_{j=1}^n f(t_j)g'(t_j)(x_j - x_{j-1}) = \sum_{j=1}^n f(t_j)(g(x_j) - g(x_{j-1})) \\
&\le \sum_{j=1}^n M_j(f)(g(x_j) - g(x_{j-1})) = U(\mathcal{P} \cup \mathcal{Q}, f, g) \le U(\mathcal{Q}, f, g)
\end{aligned}$$

and

$$\begin{aligned}
U(\mathcal{P}, fg') \geq U(\mathcal{P} \cup \mathcal{Q}, fg') &= \sum_{j=1}^{n} M_j(fg')(x_j - x_{j-1}) \\
&\geq \sum_{j=1}^{n} f(t_j)g'(t_j)(x_j - x_{j-1}) = \sum_{j=1}^{n} f(t_j)(g(x_j) - g(x_{j-1})) \\
&\geq \sum_{j=1}^{n} m_j(f)(g(x_j) - g(x_{j-1})) = L(\mathcal{P} \cup \mathcal{Q}, f, g) \geq L(\mathcal{Q}, f, g).
\end{aligned}$$

It follows that

$$\int_a^b fg' = \underline{\int_a^b} fg' \leq \overline{\int_a^b} f\,dg = \int_a^b f\,dg = \underline{\int_a^b} f\,dg \leq \overline{\int_a^b} fg' = \int_a^b fg'.$$

The theorem follows. ■

Homework:

(1) Calculate $\displaystyle\int_0^1 (x^2 + 5)\,d(2x^3 - 1)$.

(2) Calculate $\displaystyle\int_0^\pi \sin x\,d\sin x$.

(3) Let a, b, and c be real numbers for which $a < c < b$. Let f be a bounded real-valued function on $[a, b]$, and let h be an increasing function on $[a, b]$. Suppose $f \in \mathcal{R}_h[a, c]$ and $f \in \mathcal{R}_h[c, b]$. Prove that $f \in \mathcal{R}_h[a, b]$ and

$$\int_a^b f\,dh = \int_a^c f\,dh + \int_c^b f\,dh.$$

(Hint: This is similar to part (iv) of the first theorem in the previous section. You can use part (iv) of this theorem to help do this problem. Note that this problem really has nothing to do with this section; there was just more room here to include the problem.)