MATH 555 NOTES
Computing Riemann-Stieltjes Integrals

In this section, we consider various results related to the computation of Riemann-Stieltjes integrals. We sumarize several of them with our first theorem. To clarify, by $f \in \mathcal{R}_h[a,b]$ we mean that f is a bounded real-valued function on $[a,b]$, h is an increasing function on $[a,b]$, and f is Riemann-Stieltjes integrable with respect to h on $[a,b]$.

Theorem. *The following all hold:*

(i) Let f and g be in $\mathcal{R}_h[a,b]$. Then $f+g \in \mathcal{R}_h[a,b]$ and

$$\int_a^b (f+g)\,dh = \int_a^b f\,dh + \int_a^b g\,dh.$$

(ii) Let $f \in \mathcal{R}_h[a,b]$, and let c be a constant. Then $cf \in \mathcal{R}_h[a,b]$ and

$$\int_a^b cf\,dh = c\int_a^b f\,dh.$$

(iii) Suppose $f \in \mathcal{R}_{h_1}[a,b]$ and $f \in \mathcal{R}_{h_2}[a,b]$. Then $f \in \mathcal{R}_{h_1+h_2}[a,b]$ and

$$\int_a^b f\,d(h_1+h_2) = \int_a^b f\,dh_1 + \int_a^b f\,dh_2.$$

(iv) Let $f \in \mathcal{R}_h[a,b]$, and let $c \in (a,b)$. Then $f \in \mathcal{R}_h[a,c]$ and $f \in \mathcal{R}_h[c,b]$ and

$$\int_a^b f\,dh = \int_a^c f\,dh + \int_c^b f\,dh.$$

(v) Let f and g be in $\mathcal{R}_h[a,b]$. Suppose $f(x) \le g(x)$ for all $x \in [a,b]$. Then

$$\int_a^b f\,dh \le \int_a^b g\,dh.$$

(vi) Let $f \in \mathcal{R}_h[a,b]$, and let M be such that $|f(x)| \le M$ for all $x \in [a,b]$. Then $|f| \in \mathcal{R}_h[a,b]$ and

$$\left|\int_a^b f\,dh\right| \le \int_a^b |f|\,dh \le M(h(b)-h(a)).$$

Proof of (i). Let $\varepsilon > 0$, and choose partitions $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{Q}_1$, and $\mathcal{Q}_2$ such that

$$U(\mathcal{P}_1, f, h) \le \int_a^b f\,dh + \varepsilon, \quad U(\mathcal{P}_2, g, h) \le \int_a^b g\,dh + \varepsilon,$$

$$L(\mathcal{Q}_1, f, h) \geq \int_a^b f\,dh - \varepsilon, \quad \text{and} \quad L(\mathcal{Q}_2, g, h) \geq \int_a^b g\,dh - \varepsilon.$$

Then

$$\begin{aligned}
\underline{\int_a^b} (f+g)\,dh &\leq \overline{\int_a^b} (f+g)\,dh \leq U(\mathcal{P}_1 \cup \mathcal{P}_2, f+g, h) \\
&\leq U(\mathcal{P}_1 \cup \mathcal{P}_2, f, h) + U(\mathcal{P}_1 \cup \mathcal{P}_2, g, h) \leq U(\mathcal{P}_1, f, h) + U(\mathcal{P}_2, g, h) \\
&\leq \Big(\int_a^b f\,dh + \varepsilon\Big) + \Big(\int_a^b g\,dh + \varepsilon\Big) \\
&\leq L(\mathcal{Q}_1, f, h) + L(\mathcal{Q}_2, g, h) + 4\varepsilon \leq L(\mathcal{Q}_1 \cup \mathcal{Q}_2, f, h) + L(\mathcal{Q}_1 \cup \mathcal{Q}_2, g, h) + 4\varepsilon \\
&\leq L(\mathcal{Q}_1 \cup \mathcal{Q}_2, f+g, h) + 4\varepsilon \leq \underline{\int_a^b} (f+g)\,dh + 4\varepsilon.
\end{aligned}$$

The result follows by letting ε approach zero. ■

Proof of (v). The condition $f(x) \leq g(x)$ for all $x \in [a,b]$ implies that $U(\mathcal{P}, f, h) \leq U(\mathcal{P}, g, h)$ for all partitions $\mathcal{P}$ of $[a,b]$. It follows that $\overline{\int_a^b} f\,dh \leq U(\mathcal{P}, g, h)$ for all partitions $\mathcal{P}$ of $[a,b]$ and, hence, $\overline{\int_a^b} f\,dh \leq \overline{\int_a^b} g\,dh$. Since f and g are in $\mathcal{R}_h[a,b]$, the result follows. ■

Questions: If $f + g \in \mathcal{R}_h[a,b]$, then must $f \in \mathcal{R}_h[a,b]$ and (or) $g \in \mathcal{R}_h[a,b]$? If $f + g \in \mathcal{R}_h[a,b]$ and $f \in \mathcal{R}_h[a,b]$, must $g \in \mathcal{R}_h[a,b]$? How does one proof (ii)? If $f \in \mathcal{R}_h[a,b]$ and c is a constant, is it true that $f \in \mathcal{R}_{ch}[a,b]$ and

$$\int_a^b f\,d(ch) = c\int_a^b f\,dh?$$

What happens if $f(x) \geq g(x)$ in part (v)? Is M needed for the first inequality in (vi)?

Next, we turn to a special but useful case, namely when the function f is continuous and the function g is a step function.

Definitions and Notations. Let $g : [a,b] \mapsto \mathbb{R}$ be such that g is constant on (x'_{k-1}, x'_k) for some partition $\{x'_0, x'_1, \ldots, x'_m\}$ of $[a,b]$ and for every $k \in \{1, 2, \ldots, m\}$. Then g is said to be a *step function*. The *size of the steps* (or the *jumps*) are $s_k(g)$ defined by

$$s_k(g) = \lim_{\varepsilon \to 0^+} (g(x_k + \varepsilon) - g(x_k - \varepsilon)) \quad \text{for } 1 \leq k \leq m-1,$$

$$s_0(g) = \lim_{\varepsilon \to 0^+} (g(x_0 + \varepsilon) - g(x_0)), \quad \text{and} \quad s_m(g) = \lim_{\varepsilon \to 0^+} (g(x_m) - g(x_m - \varepsilon)).$$

Comment: No condition is put on the value of g at the numbers x'_k. We are, however, interested in monotone increasing step functions g which does restrict $g(x'_k)$ to be in an interval of length $s_k(g)$.

Theorem. *Let $f : [a,b] \mapsto \mathbb{R}$ be continuous, and let $g : [a,b] \mapsto \mathbb{R}$ be a monotone increasing step function. Then $f \in \mathcal{R}_g[a,b]$. Furthermore, given the notation above,*

$$\int_a^b f\,dg = \sum_{k=0}^{m} f(x'_k)s_k(g).$$

Proof. That $f \in \mathcal{R}_g[a,b]$ follows from the first theorem in the last section. By Problem (1) of the previous section, it suffices to show that there is an $\varepsilon > 0$ such that if $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a,b]$ with $\|\mathcal{P}\| < \varepsilon$, then

$$(*) \qquad L(\mathcal{P}, f, g) \le \sum_{k=0}^{m} f(x'_k)s_k(g) \le U(\mathcal{P}, f, g).$$

We take $\varepsilon = \min\{x'_k - x'_{k-1} : 1 \le k \le m\}/2$. It follows that if j is such that $x_{j-1} < x'_k < x_j$, then $s_k(g) = g(x_j) - g(x_{j-1})$. In this case,

$$m_j(f)(g(x_j) - g(x_{j-1})) \le f(x'_k)s_k(g) \le M_j(f)(g(x_j) - g(x_{j-1})).$$

These inqualities also hold in the case $k = 0$ and $j = 1$ and in the case $k = m$ and $j = n$. If on the other hand $x'_k = x_j$ for some $j \in \{1, 2, \ldots, n-1\}$, then one gets

$$\begin{aligned} m_j(f)(g(x_j) - g(x_{j-1})) + m_{j+1}(f)(g(x_{j+1}) - g(x_j)) &\le f(x'_k)s_k(g) \\ &\le M_j(f)(g(x_j) - g(x_{j-1})) + M_{j+1}(f)(g(x_{j+1}) - g(x_j)). \end{aligned}$$

We easily deduce that $(*)$ holds and, hence, the theorem. ■

Questions: Is this theorem true if f is not continuous on $[a,b]$? How might this theorem take into account certain discontinuities?

Homework:

(1) Prove part (iii) of the first theorem.

(2) Prove part (iv) of the first theorem.

(3) Prove part (vi) of the first theorem.

(4) Calculate $\displaystyle\int_0^{10} 2^x\,d[x]$ where $[x]$ denotes the greatest integer $\le x$.

(5) Let A be a set of positive integers, and let $A(x) = |\{a \in A : a \le x\}|$. Let $f : [0,\infty) \mapsto \mathbb{R}$ be continuous. Explain why, for all $x \ge 0$,

$$\int_0^x f(t)\,dA(t) = \sum_{a \in A, a \le x} f(a).$$