MATH 555 NOTES
Examples of Functions which are Riemann-Stieltjes Integrable

Corresponding to the notes dealing with examples of Riemann integrable functions, we establish a result here for functions f which are continuous. The proof should be compared to the previous proof for the analogous result for Riemann integrals.

Theorem. *Let $f : [a,b] \mapsto \mathbb{R}$ be continuous, and let $g : [a,b] \mapsto \mathbb{R}$ be monotone increasing. Then $f \in \mathcal{R}_g[a,b]$.*

Proof. If $g(a) = g(b)$, then $g(x)$ is constant and the result easily follows. Suppose now that $g(a) \neq g(b)$. Let $\varepsilon > 0$. Since f is continuous on the compact set $[a,b]$, it is uniformly continuous on $[a,b]$. Thus, there is a $\delta > 0$ such that if $|x-y| < \delta$ and x and y are in $[a,b]$ then $|f(x) - f(y)| < \varepsilon/(2(g(b)-g(a)))$. Let $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a,b]$ with $\|\mathcal{P}\| < \delta$. It follows that for each $j \in \{1,2,\ldots,n\}$, $0 \le M_j(f) - m_j(f) \le \varepsilon/(2(g(b)-g(a)))$. Thus,

$$\begin{aligned} U(\mathcal{P},f,g) - L(\mathcal{P},f,g) &= \sum_{j=1}^{n} \big(M_j(f) - m_j(f)\big)(g(x_j) - g(x_{j-1})) \\ &\le \frac{\varepsilon}{2(g(b)-g(a))} \sum_{j=1}^{n} (g(x_j) - g(x_{j-1})) = \frac{\varepsilon}{2(g(b)-g(a))}(g(b)-g(a)) < \varepsilon. \end{aligned}$$

It follows from the last theorem in the previous section that $f \in \mathcal{R}_g[a,b]$. ∎

A similar result is the following:

Theorem. *Let $f : [a,b] \mapsto \mathbb{R}$ be monotone, and let $g : [a,b] \mapsto \mathbb{R}$ be a continuous increasing function. Then $f \in \mathcal{R}_g[a,b]$.*

Proof. We suppose that f is monotone increasing; a similar argument works if f is monotone decreasing. An easy argument can be given in the case that f is constant, so we may suppose that $f(a) \neq f(b)$ and do so. Let $\varepsilon > 0$. Since g is continuous on the compact set $[a,b]$, it is uniformly continuous on $[a,b]$. Thus, there is a $\delta > 0$ such that if $|x-y| < \delta$ and x and y are in $[a,b]$ then $|g(x) - g(y)| < \varepsilon/(2(f(b)-f(a)))$. Let $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[a,b]$ with $\|\mathcal{P}\| < \delta$. The above and g increasing implies that for each $j \in \{1,2,\ldots,n\}$, $0 \le g(x_j) - g(x_{j-1}) \le \varepsilon/(2(f(b)-f(a)))$. Since f is increasing, $M_j(f) = f(x_j)$ and $m_j(f) = f(x_{j-1})$ for each $j \in \{1,2,\ldots,n\}$. Thus,

$$\begin{aligned} U(\mathcal{P},f,g) - L(\mathcal{P},f,g) &= \sum_{j=1}^{n} \big(M_j(f) - m_j(f)\big)(g(x_j) - g(x_{j-1})) \\ &\le \frac{\varepsilon}{2(f(b)-f(a))} \sum_{j=1}^{n} (f(x_j) - f(x_{j-1})) = \frac{\varepsilon}{2(f(b)-f(a))}(f(b)-f(a)) < \varepsilon. \end{aligned}$$

From the last theorem in the previous section, we deduce that $f \in \mathcal{R}_g[a,b]$. ∎

Questions: What are some examples of functions f and g with f Riemann-Stieltjes integrable with respect to g? If

$$f(x) = \begin{cases} x & \text{for } 0 \le x < 1 \\ 3 - x & \text{for } 1 \le x \le 3 \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} 0 & \text{for } 0 \le x \le 2 \\ x & \text{for } 2 < x \le 3 \end{cases},$$

then is $f \in \mathcal{R}_g[0,1]$? How might the theorems above be modified to take care of this situation?

Homework:

(1) Prove that for every $\varepsilon > 0$,

$$\begin{aligned} \inf\{U(\mathcal{P}, f, g) :& \mathcal{P} \text{ is a partition of } [a,b]\} \\ &= \inf\{U(\mathcal{P}, f, g) : \mathcal{P} \text{ is a partition of } [a,b] \text{ and } \|\mathcal{P}\| < \varepsilon\} \end{aligned}$$

and

$$\begin{aligned} \sup\{L(\mathcal{P}, f, g) :& \mathcal{P} \text{ is a partition of } [a,b]\} \\ &= \sup\{L(\mathcal{P}, f, g) : \mathcal{P} \text{ is a partition of } [a,b] \text{ and } \|\mathcal{P}\| < \varepsilon\}. \end{aligned}$$

(2) Let $f : [a,b] \mapsto \mathbb{R}$ be monotone, and let $g : [a,b] \mapsto \mathbb{R}$ be monotone increasing. Then show by example that f may not be in $\mathcal{R}_g[a,b]$.