MATH 555 NOTES
Review Problems

(1) Prove that $\sqrt{x}$ is uniformly continuous on $[0,1]$. (Hint: If x and y are in $[0,1]$, then $\sqrt{|x-y|} \leq \sqrt{x} + \sqrt{y}$. Explain why this is true and use it.)

(2) Let a, b, and c be real numbers with $a < b < c$, and suppose $f : [a,c] \mapsto \mathbb{R}$ is continuous on $[a,c]$ and twice differentiable on (a,c). The purpose of this problem is to prove that there is an $x_0 \in (a,c)$ for which

$$(*) \qquad (b-a)f(c) - (c-a)f(b) + (c-b)f(a) = \frac{1}{2}(b-a)(c-a)(c-b)f''(x_0).$$

Let

$$A = \frac{2}{(b-a)(c-a)(c-b)} \left((b-a)f(c) - (c-a)f(b) + (c-b)f(a) \right),$$

and define

$$g(x) = (b-a)f(x) - (x-a)f(b) + (x-b)f(a) - \frac{1}{2}(b-a)(x-a)(x-b)A.$$

(a) Use Rolle's Theorem (twice) to prove that there exist $x_1 \in (a,b)$ and $x_2 \in (b,c)$ such that $g'(x_1) = g'(x_2) = 0$.

(b) Explain why there exists $x_0 \in (a,c)$ such that $A = f''(x_0)$.

(c) Why does this imply $(*)$?

Comment: The formula $(*)$ above arises in the study of finite differences. It is associated with a second order divided difference.

(3) Let a and b be real numbers with $a < b$, and let $f : [a,b] \mapsto \mathbb{R}$. Then f is said to satisfy a Lipschitz condition on $[a,b]$ if there exists an $M > 0$ such that $|f(x) - f(y)| \leq M|x-y|$ for all x and y in $[a,b]$. If f has a continuous derivative on $[a,b]$, prove that f satisfies a Lipschitz condition on $[a,b]$.

(4) Let a and b be real numbers with $a < b$. Let $\mathcal{P}_k$ be a partition of $[a,b]$ for each positive integer k, and suppose that $f : [a,b] \mapsto \mathbb{R}$ is bounded. Suppose further that

$$\lim_{k \to \infty} (U(\mathcal{P}_k, f) - L(\mathcal{P}_k, f)) = 0.$$

Prove that f is Riemann integrable on $[a,b]$, $\lim_{k \to \infty} U(\mathcal{P}_k, f)$ and $\lim_{k \to \infty} L(\mathcal{P}_k, f)$ exist, and

$$\int_a^b f = \lim_{k \to \infty} U(\mathcal{P}_k, f) = \lim_{k \to \infty} L(\mathcal{P}_k, f).$$

(Hint: That f is Riemann integrable on $[a, b]$ follows directly from a theorem in class, and for the rest you might want to look at the proof of this theorem.)

(5) Find with proof the value of $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \cos\left(\frac{k\pi}{2n}\right)$. (Hint: Explain why the sum with the factor $1/n$ in front may be viewed as a multiple of a lower sum for $f(x) = \cos x$ and then use Problem (4) above. You should feel free to use results about the function $\cos x$ in this problem; for example, you may use that the antiderivative of $\cos x$ is $\sin x$ and that $\cos x$ is decreasing for $x \in [0, \pi]$.)