MATH 555 NOTES
Miscellaneous on Riemann Integrals

The following theorems we state without proof. Some will arise later as a consequence of a more general study of the Riemann-Stieltjes integral. They nevertheless are worth noting at this point. As before, we consider a and b to be real numbers below with $a < b$.

Theorem 1. *Let f and g be in $\mathcal{R}[a,b]$, and let c be a constant. Then*

(i) $f+g \in \mathcal{R}[a,b]$ *and* $\displaystyle\int_a^b (f+g) = \int_a^b f + \int_a^b g$,

(ii) $cf \in \mathcal{R}[a,b]$ *and* $\displaystyle\int_a^b cf = c\int_a^b f$, *and*

(iii) $fg \in \mathcal{R}[a,b]$.

Theorem 2. *Let $f : [a,b] \mapsto \mathbb{R}$ be bounded, and let $c \in (a,b)$. Then $f \in \mathcal{R}[a,b]$ if and only if $f \in \mathcal{R}[a,c]$ and $f \in \mathcal{R}[c,b]$. Furthermore, if $f \in \mathcal{R}[a,b]$, then*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Theorem 3. *Let $f \in \mathcal{R}[a,b]$. Then $|f| \in \mathcal{R}[a,b]$ and*

$$\left| \int_a^b f \right| \leq \int_a^b |f|.$$

Theorem 4. *Let $f : [a,b] \mapsto [c,d]$ with $f \in \mathcal{R}[a,b]$, and let $\phi \in \mathcal{R}[c,d]$ be continuous. Then $\phi \circ f \in \mathcal{R}[a,b]$.*

Comments and Questions: Theorem 1 (iii) does not give the value of $\displaystyle\int_a^b fg$ in terms of $\displaystyle\int_a^b f$ and $\displaystyle\int_a^b g$; what if anything do you recall about computing $\displaystyle\int_a^b fg$? In Theorem 2, is the condition $a < c < b$ necessary? If f is Riemann integrable on an interval containing a, b, and c, does the final equation of Theorem 2 still hold (independent of whether $a < c < b$)? When might equality hold in Theorem 3? If there are x_1 and x_2 in $[a,b]$ for which $f(x_1) < 0 < f(x_2)$, then must strict inequality hold in Theorem 3? What if in addition we require that f is continuous on $[a,b]$? Note that in Theorem 4, we have $f([a,b]) \subseteq [c,d]$ (in other words, f is bounded) and not necessarily $f([a,b]) = [c,d]$. What kind of examples can you think of for Theorem 4?

Homework:

(1) Without using the results above but using any result previously established in the notes, prove that if a, b, and c are real numbers with $a < c < b$ and if $f \in \mathcal{R}[a,c]$ and

$f \in \mathcal{R}[c, b]$, then $f \in \mathcal{R}[a, b]$. (Possible Approach: Use the first theorem from the section "Examples of Functions which are Riemann Integrable".)

(2) Suppose $|f| \in \mathcal{R}[a, b]$. Prove or disprove that $f \in \mathcal{R}[a, b]$.

(3) Explain how Theorem 4 and Parts (i) and (ii) of Theorem 1 can be used to establish Part (iii) of Theorem 1.