# MATH 555 NOTES
## The Fundamental Theorem of Calculus

**Definition.** Let $f : [a, b] \mapsto \mathbb{R}$. We say that a function $F : [a, b] \mapsto \mathbb{R}$ is an *antiderivative* of $f$ on $[a, b]$ if $F' = f$ on $[a, b]$.

**Question:** If $F$ and $G$ are both antiderivatives of a function $f$ on $[a, b]$, then how are $F$ and $G$ related?

**Theorem (The Fundamental Theorem of Calculus).** *If $f \in \mathcal{R}[a, b]$ and $F$ is an antiderivative of $f$ on $[a, b]$, then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

*Proof.* Let $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ be an arbitrary partition of $[a, b]$. By the Mean Value Theorem, for each $j \in \{1, 2, \ldots, n\}$, there is a $t_j \in (x_{j-1}, x_j)$ such that

$$F(x_j) - F(x_{j-1}) = F'(t_j)(x_j - x_{j-1}) = f(t_j)(x_j - x_{j-1}).$$

Observe that

$$L(\mathcal{P}, f) \leq \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}) \leq U(\mathcal{P}, f)$$

and

$$\sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}) = \sum_{j=1}^{n} \big(F(x_j) - F(x_{j-1})\big) = F(b) - F(a).$$

We deduce that

$$\underline{\int_a^b} f \leq F(b) - F(a) \leq \overline{\int_a^b} f.$$

Since $f \in \mathcal{R}[a, b]$, $\underline{\int_a^b} f = \overline{\int_a^b} f = \int_a^b f$ and the result follows. ■

**Comments and Questions:** The Fundamental Theorem of Calculus (due to Cauchy) tells us that we can compute the area of a region under a curve $y = f(x)$ and above the $x$-axis by computing an antiderivative of $f$. It thus connects the notions of area and differentiation (or antidifferentiation). What kind of homework exercise could you make up that would require a student to use the Fundamental Theorem of Caluculus? To use the Fundamental Theorem of Caluculus, does one need the function $f$ to be continuous?

Can you think of other variations of the Fundamental Theorem of Calculus (think in terms of $F(x) = \int_a^x f(t)dt$; for this one typically requires $f$ to be continuous on $[a, b]$)?

**Homework:**

(1) If $f : [a, b] \mapsto \mathbb{R}$ is not bounded on $[a, b]$, then we have not defined $\int_a^b f$. We have not even defined $\overline{\int_a^b} f$ and $\underline{\int_a^b} f$. Suppose $f$ is bounded on $(t, b]$ for each $t \in (a, b]$. We define

$$\int_a^b f = \lim_{t \to a} \int_t^b f$$

provided $\int_t^b f$ exists for each $t \in (a, b]$ and the limit on the right-hand side above exists. Using this idea and theorems on Riemann integrals, determine with proof the value of $\int_0^1 \frac{1}{\sqrt{x}}\, dx$.