MATH 555 NOTES
Some Review Material

Question: How does one (usually) show that two sets are equal?

Definition. A sequence $\{p_n\}$ *converges* (to p) provided that for some p and for every $\varepsilon > 0$, there is an N such that if $n \geq N$, then $|p_n - p| < \varepsilon$.

Questions: How does one phrase this in terms of ε-neighborhoods? What notation is associated with this definition? What does diverge mean? What theorems related to convergent sequences can you remember? How are convergent sequences related to bounded sequences?

Definition. A sequence $\{p_n\}$ is *Cauchy* provided that for every $\varepsilon > 0$, there is an N such that if both n and m are $\geq N$, then $|p_n - p_m| < \varepsilon$.

Theorem. *A sequence $\{p_n\}$ converges if and only if it is Cauchy.*

Questions: If they're the same, why study both of these concepts? What might be advantages or disadvantages of each?

Definition. A point p is a *limit point* of a set S provided that every ε-neighborhood of p contains a point q satisfying $q \in S$ and $q \neq p$.

Theorem (Bolzano-Weierstrass). *Every bounded infinite set in $\mathbb{R}$ has a limit point.*

Questions: What are the limit points for $\mathbb{Z}$, for $\mathbb{Q}$, and for $(0,1]$? Is the condition in the theorem that the set is bounded necessary? (How does one explain the answer to such a question?) Is the condition necessary if the set is uncountable?

Definition. A set S is *compact* provided that every open covering of S contains a finite subcovering.

Questions: What is an open covering? What does it mean for a set to be open? What does it mean for a set to be closed? Is a set either open or closed? What are examples of open and closed sets? Which if any of the sets $\emptyset$, $(0,1]$, $[0,\infty)$, $[-10,5]$, and $\mathbb{R}$ is compact?

Theorem. *A set $S \subseteq \mathbb{R}$ is compact if and only if it is closed and bounded.*

Questions: What are the advantages of this result over the definition of a set being compact? What are the disadvantages?

Definition. A function f *has a limit L as x approaches x_0* provided that for every $\varepsilon > 0$, there is a $\delta > 0$ such that if $0 < |x - x_0| < \delta$ and x is in the domain of f, then $|f(x) - L| < \varepsilon$.

Questions: Must x_0 be in the domain of f? What condition should be added on x_0 in this definition? What's the notation associated with this definition?

Definition. A function f *is continuous at x_0* provided that $\lim\limits_{x \to x_0} f(x) = f(x_0)$.

Question: How does one express the definitioin in terms of ε and δ?

Theorem. *Let K be a compact set, and suppose $f : K \mapsto \mathbb{R}$ is a continuous function. Then $\max\{f(x) : x \in K\}$ and $\min\{f(x) : x \in K\}$ exist.*

Questions: What does this mean? Is the theorem true even if K is not compact? Is the theorem true even if f is not continuous?

Homework:

(1) Let A and B be sets in the range of a function f. Prove that $f^{-1}(A) - f^{-1}(B) = f^{-1}(A - B)$.

(2) Prove that $[-1, 1)$ is not compact by using the definition of a compact set (to get credit for the problem, use the definition and not any theorems about compact sets).

(3) What is an *interior* point? Prove that $1/6$ is an interior point of $(0, 2]$.

(4) Let $a_1 = \sqrt{6}$ and $a_{n+1} = \sqrt{6 + a_n}$ for $n \geq 1$.

(a) Show that $a_n \leq 3$ for all $n \geq 1$.

(b) Show that $\{a_n\}_{n=1}^{\infty}$ is an *increasing* sequence.

(c) Explain why $\{a_n\}_{n=1}^{\infty}$ converges.

(d) Determine the value of $\lim_{n \to \infty} a_n$.

(5) Complete the table below indicating $\mathrm{Int}(E)$ (the interior of E), the set of isolated points of E, and whether E is open, closed, both, or neither. An answer of "open" or "closed" in the last column will mean that you think E is "open and not closed" or "closed and not open" respectively. You do not need to show work on this problem.

E	$\mathrm{Int}(E)$	isolated pts. of E	open/closed?
$(-1, 4]$			
$(-\infty, 1] \cap \mathbb{Q}$			
$[2, \infty)$			
$\{\sqrt{n} : n \in \mathbb{Z}^+\}$			
$\mathbb{R}$			

(6) Suppose p is a limit point for two sets A and B.

(a) Must p be a limit point for $A \cup B$? Justify your answer.

(b) Must p be a limit point for $A \cap B$? Justify your answer.

(7) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Suppose that for every $\varepsilon > 0$, there is an n_0 such that if $n \geq n_0$, then $|a_{n+1996} - a_n| < \varepsilon$. Either prove that $\{a_n\}_{n=1}^{\infty}$ converges (if it is the case) or give an example of a sequence $\{a_n\}_{n=1}^{\infty}$ satisfying the above but for which $\{a_n\}_{n=1}^{\infty}$ does not converge (justify your example; explain why it satisfies the above property and why it does not converge).