MATH 532, 736I: MODERN GEOMETRY

Name ______________________________________

Practice Test #2

(1) Theorems are listed on the last page of this test. They may or may not have the numbering that you are accustomed to them having from class. Prove Theorem 3 using Theorem 1 (but not Theorem 2 or Theorem 4).

(2) Let A, B, and C be 3 noncollinear points. Let D be the intersection of the (extended) altitudes of ΔABC. Let M_A be the midpoint of $\overline{BC}$, and let Q_A be the midpoint of $\overline{AD}$. Let $N = (A+B+C+D)/4$. Prove that the distance from N to M_A is the same as the distance for N to Q_A. This is part of the 9-point circle theorem, so you should not make use of the 9-point circle theorem in doing this problem.

(3) Let $A = (1, -1)$ and $B = (1, 3)$. Calculate each of the following, and put the answer in the appropriate box. Each answer should be either a specific number (write it down) or a specific point (write down its coordinates). The vector parts are meant to be direct computations. The translation and rotation parts can be done easily by interpreting the problems geometrically. (You can however choose to do these problems in a different manner).

(a) $B + 2A = \boxed{}$

(b) $(B - A)^2 = \boxed{}$

(c) $T_B(A) = \boxed{}$

(d) $R_{\pi,A}(B) = \boxed{}$

(4) The function $f(x, y)$ is defined as follows. First, f rotates (x, y) about the point $A = (0, 1)$ by π, then it takes the result and rotates it about $B = (1, 1)$ by $\pi/2$, and then it takes the result and rotates it about the point $C = (2, 1)$ by $\pi/2$. Thus, we can view f as being $R_{\pi/2,C}R_{\pi/2,B}R_{\pi,A}$. Decide whether f is a translation or a rotation. If f is a translation, express f in the form $T_{(a,b)}$ where a and b are explicit numbers. If f is a rotation, express f in the form $R_{\phi,(a,b)}$ where ϕ, a, and b are explicit numbers.

$f = \boxed{}$ (either $T_{(a,b)}$ or $R_{\phi,(a,b)}$ but with specific numbers for a, b, and possibly ϕ)

(5) Let ΔABC be a triangle. Let P be a point along side $\overline{AB}$ that is midway between point B and the midpoint of $\overline{AB}$. Let Q be a point along side $\overline{BC}$ that is midway between point B and the midpoint of $\overline{BC}$. Let R be a point along side $\overline{AC}$ that is midway between point A and the midpoint of $\overline{AC}$. (See the first picture on the second to the last page.) Explain why the midpoint of $\overline{AQ}$ is the same point as the midpoint of $\overline{PR}$. (Be sure to justify any statements made that are not results done in class.)

(6) Let A, B, and C be 3 noncollinear points, and let A', B', and C' be 3 noncollinear points with $A \neq A'$, $B \neq B'$, and $C \neq C'$. Suppose that the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ intersect at the point X. Suppose further that $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ are parallel and that $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ are parallel. (See the second picture on the second to the last page.) The next two pages contain a proof that $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ are parallel. Complete the proof by filling in the boxes. There may be more than one correct way to fill in a box. The goal is to end up with a correct proof that $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ are parallel.

Proof: Since $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ intersect at X, Theorem 1 (from the last page of this exam) implies that there are real numbers k_1, k_2, and k_3 such that

$$(*) \qquad X = (1-k_1)A' + k_1 A = (1-k_2)B' + k_2 B = (1-k_3)C' + k_3 C.$$

From $(*)$, we obtain

$$k_1 A - k_2 B = (1-k_2)B' - (1-k_1)A'.$$

Next, we explain why $k_1 = \boxed{}$. Assume otherwise. Then, dividing by $k_1 - k_2$, we obtain

$$\left(\frac{k_1}{k_1-k_2}\right) A + \left(\frac{-k_2}{k_1-k_2}\right) B = \left(\frac{1-k_2}{k_1-k_2}\right) B' + \left(\frac{k_1-1}{k_1-k_2}\right) A'.$$

By Theorem 1 (from the last page of this exam) with $t = \boxed{}$, we see that the expression on the left above is a point on line $\overleftrightarrow{AB}$. By Theorem 1 (from the last page of this exam) with $t = \boxed{}$, we see that the expression on the right above is a point on line $\overleftrightarrow{A'B'}$. This is a contradiction since there is no point on both $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ (these lines are parallel). Thus, our assumption is wrong, and $k_1 = \boxed{}$.

Similarly, we deduce from $(*)$ that

$$k_2 B - k_3 C = (1-k_3)C' - (1-k_2)B'.$$

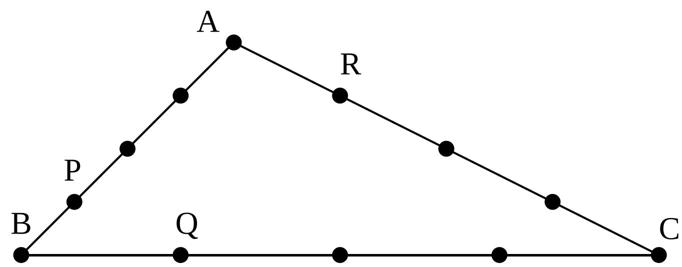
Using this equation and the fact that $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ are parallel, we obtain $k_2 = \boxed{}$. It follows that $k_1 = \boxed{}$ as well. Since $(*)$ also implies

$$k_1 A - k_3 C = \boxed{},$$

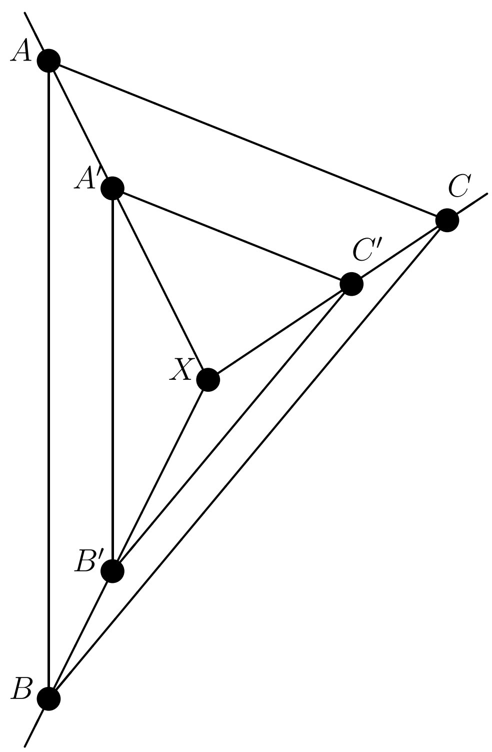
we obtain

$$\boxed{} = (1 - k_1)(C' - A').$$

We deduce that the vector $\boxed{}$ and the vector $\overrightarrow{A'C'}$ are parallel. Hence, lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ are parallel, completing the proof.



Picture for Problem 5



Picture for Problem 6

INFORMATION PAGE

Theorem 1: *Let A and B be distinct points. Then C is a point on line $\overleftrightarrow{AB}$ if and only if there is a real number t such that*

$$C = (1-t)A + tB.$$

Theorem 2: *If A, B, and C are collinear, then there are real numbers x, y, and z not all 0 such that*

$$x + y + z = 0 \quad \textit{and} \quad xA + yB + zC = \overrightarrow{0}.$$

Theorem 3: *If A, B, and C are points and there are real numbers x, y, and z not all 0 such that*

$$x + y + z = 0 \quad \textit{and} \quad xA + yB + zC = \overrightarrow{0},$$

then A, B, and C are collinear.

Theorem 4: *If A, B, and C are not collinear and if there are real numbers x, y, and z such that*

$$x + y + z = 0 \quad \textit{and} \quad xA + yB + zC = \overrightarrow{0},$$

then $x = y = z = 0$.

$$T_{(a,b)} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_{\theta,(x_1,y_1)} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & x_1(1-\cos(\theta)) + y_1\sin(\theta) \\ \sin(\theta) & \cos(\theta) & -x_1\sin(\theta) + y_1(1-\cos(\theta)) \\ 0 & 0 & 1 \end{pmatrix}$$