# MATH 532, 736I: MODERN GEOMETRY

**Test 2, , Spring 2019** Name Solutions

Show All Work

**Instructions:** This test consists of 3 pages (including an information section at the end). Put your name at the top of this page and at the top of the first page of the packet of blank paper. Work each problem in the packet of paper unless it is indicated that you can or are to do the work below. Fill in the boxes below with your answers. Show ALL of your work. Do NOT use a calculator.

$\boxed{\text{21 pts}}$ (1) Let $A = (0, 0)$ and $B = (12, 0)$. Let $P$, $Q$, and $S$ be 3 other points.

(a) If $R_{\pi,P}(A) = B$, then $P = \boxed{(6, 0)}$. (Write a point with specific coordinates.)

We obtain this by noting that $R_{\pi,P}(A) = B$ indicates that $P$ is the midpoint between $A$ and $B$, which we can calculate by $((x_A + x_B)/2, (y_A + y_B)/2)$ where $x_A$ and $y_A$ are the $x$ and $y$ coordinates of point $A$, and $x_B$ and $y_B$ are the $x$ and $y$ coordinates of point $B$.

(b) If $R_{\pi/2,Q}(B) = A$, then $Q = \boxed{(6, -6)}$. (Write a point with specific coordinates.)

We are looking for a $90°$ counterclockwise rotation that moves $B$ at $A$. Keep in mind that the center of rotation is always on the perpendicular bisector of the line segment $\overline{AB}$. Since the rotation is by $\pi/2$ counterclockwise, the segment $\overline{AB}$ is the diagonal of a square with $Q$ a vertex of the square lying below $\overline{AB}$ on the perpendicular bisector of segment $\overline{AB}$. If $M$ is the midpoint of $\overline{AB}$, then $\triangle QMA$ will be an isosceles right triangle so that $Q$ is 6 units below $M = (6, 0)$. This gives the answer shown.

(c) If $R_{\pi/3,S}(A) = B$, then $S = \boxed{(6, 6\sqrt{3})}$. (Write a point with specific coordinates.)

Again, we set $M = (6, 0)$, the midpoint of segment $\overline{AB}$. As before (and always), the point $S$ lies on the perpendicular bisector of segment $\overline{AB}$. Since the rotation is counterclockwise and takes $A$ to $B$, we see that $S$ is above $M$. Observe that $\triangle AMS$ is such that $\angle M = 90^\circ$ and $\angle S = 30^\circ$ (so $\angle A = 60^\circ$). It follows that the distance from $M$ to $S$ is $6\sqrt{3}$, giving the answer provided.

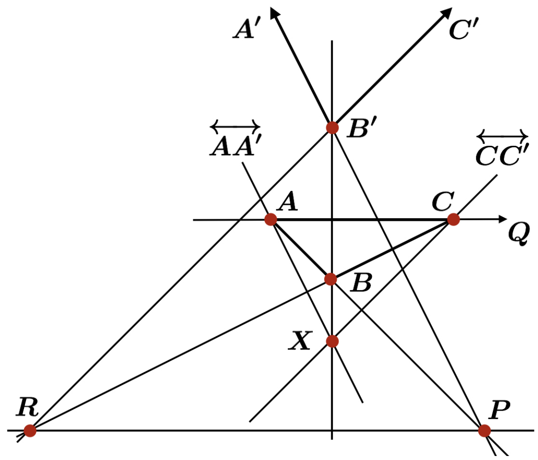
16 pts

(2) Give a proof of Theorem 3 stated on the last page of this test. Note the numbering of the theorems on the last page of this test may not be the same as in class. You are proving Theorem 3 as stated on this last page, and you should refer to the numbers of the theorems stated on this last page when doing the proof.

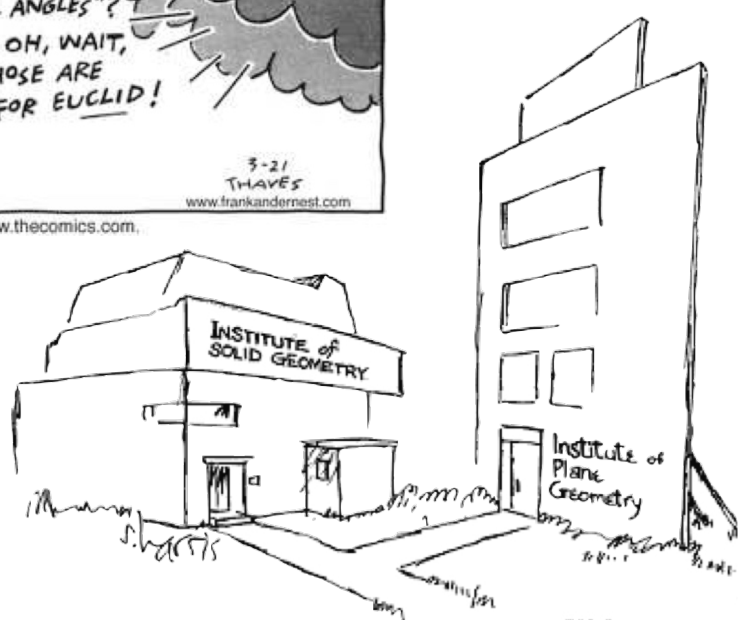
Proof. Let $A$, $B$ and $C$ be collinear points. If $A = B$, we take $x = 1$, $y = -1$ and $z = 0$. Observe that $x$ and $y$ are nonzero, $x + y + z = 0$ and $xA + yB + zC = A - B = \overrightarrow{0}$. So in this case we are done. Now suppose $A \neq B$. From Theorem 1, since the points are collinear, we get $C = (1 - t)A + tB$ for some $t$. Taking $x = 1 - t$, $y = t$, and $z = -1$, we see that $z \neq 0$, that $x + y + z = (1 - t) + t + (-1) = 0$ and, since $C = (1 - t)A + tB$, that $xA + yB + zC = (1 - t)A + tB + (-1)C = \overrightarrow{0}$. ■

24 pts

(3) In the picture, $A'$, $C'$ and $Q$ are points at infinity next to lines that pass through them. Thus, $\overleftrightarrow{PB'}$ passes through $A'$, $\overleftrightarrow{RB'}$ passes through $C'$, and $\overleftrightarrow{CA}$ passes through $Q$. The line $\overleftrightarrow{XA}$ is parallel to $\overleftrightarrow{PB'}$, so it also passes through $A'$. The line $\overleftrightarrow{XC}$ is parallel to $\overleftrightarrow{RB'}$, so it also passes through $C'$. Thus, in the projective plane (with points at infinity) $\overleftrightarrow{AA'} = \overleftrightarrow{XA}$ and $\overleftrightarrow{CC'} = \overleftrightarrow{XC}$. Since $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ all pass through $X$, we see that $\triangle ABC$ and $\triangle A'B'C'$ are perspective from the point $X$. The next page contains a proof that $\triangle ABC$ and $\triangle A'B'C'$ are perspective from a line. In other words, $P$, $Q$ and $R$ are collinear. This is the same as saying $\overleftrightarrow{PR}$ is parallel to $\overleftrightarrow{CA}$. Fill in the boxes on the next page to complete the proof.





**Note:** Four theorems are listed at the end of the quiz. Use the numbering given there.

**Proof:** The point $B$ is on lines $\overleftrightarrow{B'X}$, $\overleftrightarrow{PA}$ and $\overleftrightarrow{RC}$. Hence, by Theorem $\boxed{1}$, there are real numbers $k_1$, $k_2$, and $k_3$ with

$$(*) \qquad B = (1-k_1)B' + k_1X = (1-k_2)P + k_2A = (1-k_3)R + k_3C.$$

From $(*)$, we obtain

$$k_1X - k_2A = (1-k_2)P - (1-k_1)B'.$$

Next, we explain why $k_1 = \boxed{k_2}$. Assume otherwise. Then, dividing by $k_1 - k_2$, we obtain

$$\left(\frac{k_1}{k_1-k_2}\right)X + \left(\frac{-k_2}{k_1-k_2}\right)A = \left(\frac{1-k_2}{k_1-k_2}\right)P + \left(\frac{k_1-1}{k_1-k_2}\right)B'.$$

By Theorem 1 (from the last page of this exam) with $t = \boxed{\dfrac{-k_2}{k_1-k_2}}$, we see that the expression on the left above is a point on line $\overleftrightarrow{XA}$. By Theorem 1 (from the last page of this exam) with $t = \boxed{\dfrac{k_1-1}{k_1-k_2}}$, we see that the expression on the right above is a point on line $\overleftrightarrow{PB'}$. This is a contradiction since $\boxed{\text{line } \overleftrightarrow{XA} \text{ is parallel to line } \overleftrightarrow{PB'}}$. Thus, our assumption is wrong, and $k_1 = \boxed{k_2}$. Similarly, we deduce from $(*)$ that

$$k_1X - k_3C = (1-k_3)R - (1-k_1)B'.$$

Using this equation and the fact that $\overleftrightarrow{XC}$ and $\overleftrightarrow{RB'}$ are $\boxed{\text{parallel}}$, we obtain $k_1 = \boxed{k_3}$. It follows that $k_2 = \boxed{k_3}$ as well. Since $(*)$ also implies

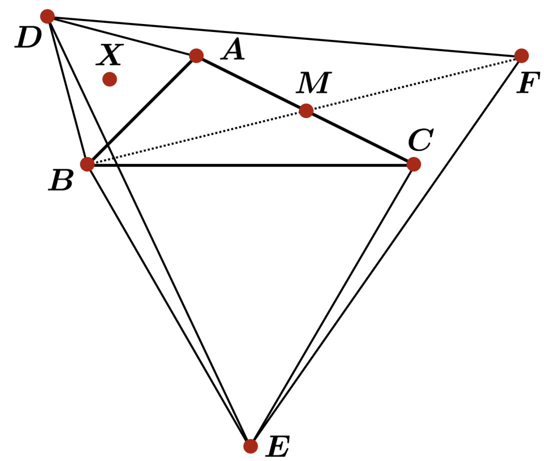
$$k_2A - k_3C = (1-k_3)R - (1-k_2)P$$

we obtain

$$\boxed{k_2(A-C)} = (1-k_2)(R-P).$$

We deduce that the vector $\boxed{\overrightarrow{CA}}$ and the vector $\overrightarrow{PR}$ are parallel. Hence, lines $\boxed{\overleftrightarrow{CA}}$ and $\overleftrightarrow{PR}$ are parallel, completing the proof. ■

$\boxed{\text{23 pts}}$ (4) In the figure, $A$, $B$ and $C$ are vertices of a triangle. The points $D$, $E$ and $F$ satisfy that $\triangle DBA$, $\triangle ECB$ and $\triangle EFD$ are equilateral triangles. The point $M$ is the midpoint of $\overline{AC}$. Prove that $M$ is also the midpoint of $\overline{BF}$. Make sure to indicate what happens with each rotation you use to do this problem. (Hint: Let $X$ be the center of the equilateral triangle $\triangle DBA$ as shown and consider




$$f = R_{2\pi/3,X} R_{\pi/3,E}.$$

Figure out whether $f$ is a rotation or a translation and be precise. Then determine how that gives you what you want. Don't forget to explain what happens each time you make a rotation for each rotation in the definition of $f$.)

Proof: We note, by Theorem 2, that $f = R_{\pi,Z}$ for some point $Z$. In order to figure out what $Z$ is, we observe that

$$R_{\pi/3,E}(C) = B \qquad \text{and} \qquad R_{2\pi/3,X}(B) = A.$$

Thus, $f(C) = A$. Since $f = R_{\pi,Z}$, we see that $R_{\pi,Z}(C) = A$ so $Z = M$, the midpoint of $\overline{AC}$. Therefore,

$$f = R_{\pi,M}.$$

Now, since

$$R_{\pi/3,E}(F) = D \qquad \text{and} \qquad R_{2\pi/3,X}(D) = B,$$

we deduce $f(F) = B$. Since $f = R_{\pi,M}$, we obtain $R_{\pi,M}(F) = B$. This implies that a rotation by $\pi$ about $M$ takes $F$ to $B$ so that $M$ is the midpoint of $\overline{BF}$, completing the proof. ■

$\boxed{\text{16 pts}}$ (5) (Note that you don't need to do the previous problem to do this one.) In the problem above, the point $M$ is the midpoint of $\overline{AC}$, and (as shown there) it is also the midpoint of $\overline{BF}$. You know something about how to tackle problems with midpoints; this problem is one of those. Explain why the lines $\overleftrightarrow{BA}$ and $\overleftrightarrow{CF}$ (not drawn) are parallel and why the segments $\overline{BA}$ and $\overline{CF}$ (not drawn) have equal length. In other words, explain why the vectors $\overrightarrow{BA}$ and $\overrightarrow{CF}$ are equal.

Explanation: Since $M$ is the midpoint of both $\overline{AC}$ and $\overline{BF}$, we have

$$\frac{A+C}{2} = M = \frac{B+F}{2}.$$

Multiplying by 2 gives us $A + C = B + F$. Subtracting $B + C$ from both sides, we obtain $A - B = F - C$ which is equivalent to $\overrightarrow{BA} = \overrightarrow{CF}$.

# INFORMATION SECTION

**Theorem 1:** *Let $A$ and $B$ be distinct points. Then $C$ is a point on line $\overleftrightarrow{AB}$ if and only if there is a real number $t$ such that $C = (1-t)A + tB$.*

**Theorem 2:** *Let $\alpha_1, \ldots, \alpha_n$ be real numbers (not necessarily distinct), and let $A_1, \ldots, A_n$ and $B_1, \ldots, B_k$ be points (not necessarily distinct). Let $f$ be a product of the $n$ rotations $R_{\alpha_j, A_j}$ and the $k$ translations $T_{B_j}$ with each of the $n$ rotations and $k$ translations occurring exactly once in the product. If $\alpha_1 + \cdots + \alpha_n$ is not an integer multiple of $2\pi$, then there is point $C$ such that*

$$f = R_{\alpha_1+\alpha_2+\cdots+\alpha_n, C}.$$

*If $\alpha_1 + \cdots + \alpha_n$ is an integer multiple of $2\pi$, then $f$ is a translation.*

**Theorem 3:** *If $A$, $B$ and $C$ are collinear, then there are real numbers $x$, $y$ and $z$ not all 0 such that*

$$x + y + z = 0 \quad \textit{and} \quad xA + yB + zC = \overrightarrow{0}.$$

**Theorem 4:** *If $A$, $B$ and $C$ are points and there are real numbers $x$, $y$, and $z$ not all 0 such that*

$$x + y + z = 0 \quad \textit{and} \quad xA + yB + zC = \overrightarrow{0},$$

*then $A$, $B$ and $C$ are collinear.*

$$T_{(a,b)} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \quad R_{\theta,(x_1,y_1)} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & x_1(1-\cos(\theta)) + y_1\sin(\theta) \\ \sin(\theta) & \cos(\theta) & -x_1\sin(\theta) + y_1(1-\cos(\theta)) \\ 0 & 0 & 1 \end{pmatrix}$$