MATH 532, 736I: MODERN GEOMETRY

Test 2, Spring 2016 Name ______________________

Show All Work

Instructions: This test consists of 4 pages (including an information section at the end). Put your name at the top of this page and at the top of the first page of the packet of blank paper. Work each problem in the packet of paper unless it is indicated that you can or are to do the work below. Fill in the boxes below with your answers. Show <u>ALL</u> of your work. Do <u>NOT</u> use a calculator.

18 pts (1) Let $A = (0, 0)$ and $B = (2016, 0)$. Let P, Q, and T be 3 other points.

(a) If $R_{\pi,P}(A) = B$, then $P =$. (Give me a specific point.)

(b) If $R_{\pi/2,Q}(A) = B$, then $Q =$. (Give me a specific point.)

(c) If $R_{\pi/2,T}(B) = A$, then $T =$. (Give me a specific point.)

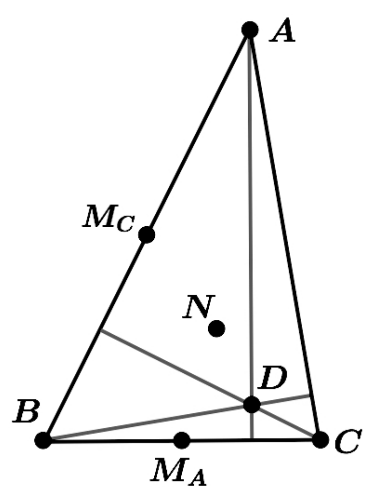
16 pts (2) Let $A = (1, 1)$ and $B = (0, 2016)$. Let $f = R_{\pi/2,B}R_{\pi/2,A}$. Then a theorem on the last section allows us to deduce that either $f = T_P$ for some point P or $f = R_{\phi,P}$ for some angle ϕ and point P. Determine which it is, and find the point P if it is a translation and the angle ϕ and point P if it is a rotation. (There is more than one way to do this problem, but using matrices is likely to lead to more partial credit if you do something wrong.)

$f =$

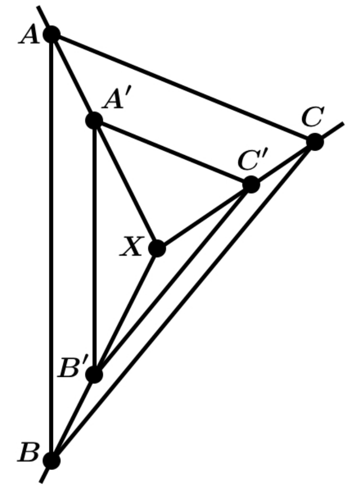
16 pts (3) Let A, B, and C be 3 noncollinear points. Let M_A be the midpoint of $\overline{BC}$, and let M_C be the midpoint of $\overline{AB}$. Let D be the intersection of the (extended) altitudes of ΔABC. Let

$$N = \frac{A + B + C + D}{4}.$$

Prove that the distance from N to M_A is the same as the distance from N to M_C. This is part of the 9-point circle theorem, so you should not make use of the 9-point circle theorem in doing this problem. Instead you want to prove this part of the theorem.



25 pts (4) Let A, B, and C be 3 noncollinear points, and let A', B', and C' be 3 noncollinear points with $A \neq A'$, $B \neq B'$, and $C \neq C'$. Suppose that the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ intersect at the point X. Suppose further that $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ are parallel and that $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ are parallel. Below is a proof that $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ are parallel. Complete the proof by filling in the boxes. There may be more than one correct way to fill in a box. The goal is to end up with a correct proof that $\overleftrightarrow{AC}$ and $\overleftrightarrow{A'C'}$ are parallel.



Proof: Since $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$, and $\overleftrightarrow{CC'}$ intersect at X, Theorem 1 (from the last section of this exam) implies that there are real numbers k_1, k_2, and k_3 such that

(1) $$X = (1 - k_1)A' + k_1 A = (1 - k_2)B' + k_2 B = (1 - k_3)C' + k_3 C.$$

From (1), we obtain

(2) $$k_1 A - k_2 B = (1 - k_2)B' - (1 - k_1)A'.$$

Next, we explain why

(3) $$k_1 = \boxed{}.$$

Assume otherwise. Then, dividing both sides of (2) by $k_1 - k_2$, we obtain

$$\left(\frac{k_1}{k_1 - k_2}\right) A + \left(\frac{-k_2}{k_1 - k_2}\right) B = \boxed{}$$

By Theorem 1 (from the last section of this exam) with $t = \boxed{}$, we see that the expression on the left above is a point on line $\overleftrightarrow{AB}$. By Theorem 1 (from the last section of this exam) with $t = \boxed{}$, we see that the expression on the right above is a point on line $\overleftrightarrow{A'B'}$. This is a contradiction since there is no point on both $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ (these lines are parallel). Thus, our assumption is wrong, and (3) follows. From (1), we also have

$$k_2 B - k_3 C = (1 - k_3)C' - (1 - k_2)B'.$$

Using this equation and the fact that $\overleftrightarrow{BC}$ and $\overleftrightarrow{B'C'}$ are parallel, we obtain, in a similar manner to what was done on the previous page to obtain (3), that

$$k_2 = \boxed{\quad\quad}. \tag{4}$$

Combining (3) and (4), we deduce

$$k_3 = \boxed{\quad\quad}. \tag{5}$$

Since (1) also implies

$$k_1 A - k_3 C = \boxed{\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad},$$

we obtain from (5) that

$$\boxed{\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad} = (1 - k_1)(C' - A').$$

(Only use one of k_1, k_2 and k_3 above.)

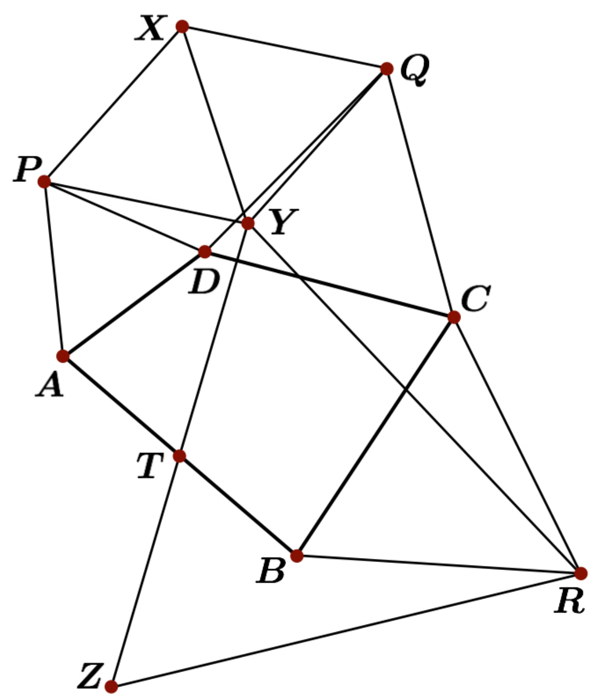
We deduce that the vector $\boxed{\quad\quad}$ and the vector $\overrightarrow{A'C'}$ are parallel. Hence, lines $\boxed{\quad\quad}$ and $\overleftrightarrow{A'C'}$ are parallel, completing the proof. ■

25 pts

(5) In the figure, A, B, C and D are vertices of a quadrilateral. The six triangles $\triangle PAD$, $\triangle QDC$, $\triangle RCB$, $\triangle PXY$, $\triangle QXY$ and $\triangle RYZ$ are all equilateral. The point T is the intersection of $\overline{AB}$ and $\overline{YZ}$. The goal of the problem is to show that T is the midpoint of both $\overline{AB}$ and $\overline{YZ}$. To do so, set

$$f = R_{\pi,M} R_{\pi/3,R} R_{\pi/3,Q} R_{\pi/3,P}$$

where M is the midpoint of $\overline{AB}$. Note: we do NOT initially know that $M = T$.



(a) Calculate $f(A)$. It should be a point in the picture. Put your answer below.

$$f(A) = \boxed{\quad\quad}$$

(b) From a theorem from the last section, f is either a translation or a rotation. Explain clearly what f is. As this is needed for part (c), a precise answer should be given here.

f is

Explanation for your answer:

(c) By looking at the value of f at another point, explain why $M = T$ and why T is the midpoint of $\overline{YZ}$. (Hint: Do NOT look at $f(P)$. Find some other point that does the job.)

INFORMATION SECTION

Theorem 1: *Let A and B be distinct points. Then C is a point on line $\overleftrightarrow{AB}$ if and only if there is a real number t such that $C = (1-t)A + tB$.*

Theorem 2: *Let $\alpha_1, \dots, \alpha_n$ be real numbers (not necessarily distinct), and let $A_1, \dots, A_n$ and $B_1, \dots, B_k$ be points (not necessarily distinct). Let f be a product of the n rotations R_{α_j, A_j} and the k translations T_{B_j} with each of the n rotations and k translations occurring exactly once in the product. If $\alpha_1 + \cdots + \alpha_n$ is not an integer multiple of 2π, then there is point C such that*

$$f = R_{\alpha_1+\alpha_2+\cdots+\alpha_n, C}.$$

If $\alpha_1 + \cdots + \alpha_n$ is an integer multiple of 2π, then f is a translation.

$$T_{(a,b)} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \quad R_{\theta,(x_1,y_1)} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & x_1(1-\cos(\theta)) + y_1\sin(\theta) \\ \sin(\theta) & \cos(\theta) & -x_1\sin(\theta) + y_1(1-\cos(\theta)) \\ 0 & 0 & 1 \end{pmatrix}$$