NOTES ON TRANSLATIONS AND ROTATIONS

We associate with each point (x, y) the column $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ which we will sometimes write as $(x, y, 1)^T$. A *translation* of the Euclidean plane is a function f which maps each point (x, y) to $(x + a, y + b)$ for some real numbers a and b. To make matters more precise, we shall refer to f as a translation by (a, b). We may view such a translation as mapping $(x, y, 1)^T$ into $(x + a, y + b, 1)^T$. Since

$$\begin{pmatrix} x+a \\ y+b \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix},$$

we may therefore think of f as simply being multiplication by the matrix above. We shall refer to the above matrix as $T_{(a,b)}$. If P represents the point (a, b), we will sometimes write T_P. Thus,

$$T_P = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

represents a translation of the Euclidean plane by P. If $P = (0, 0)$, observe that T_P maps each point to itself. In this case, we will call T_P the identity transformation.

Now, consider a point $A = (x_1, y_1)$ and a real number ϕ. A *rotation* of the Euclidean plane about A by an angle ϕ is a function f which maps each point $B = (x, y)$ to $C = (x', y')$ where C is the same distance as B from A and where the angle measured counterclockwise from the vector $\overrightarrow{AB}$ to the vector $\overrightarrow{AC}$ is ϕ. It will be convenient to also find a matrix representation of such a rotation. Suppose for the moment that $A = (0, 0)$. We can write B in polar coordinates as (r, θ). Then C has the polar coordinate representation $(r, \theta + \phi)$. Hence,

$$x' = r\cos(\theta + \phi) = r\cos(\theta)\cos(\phi) - r\sin(\theta)\sin(\phi) = x\cos(\phi) - y\sin(\phi)$$

and

$$y' = r\sin(\theta + \phi) = r\cos(\theta)\sin(\phi) + r\sin(\theta)\cos(\phi) = x\sin(\phi) + y\cos(\phi).$$

In matrix notation, we may combine these as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

In general, with $A = (x_1, y_1)$, we may obtain (x', y') by translating the Euclidean plane first by $(-x_1, -y_1)$, and then performing the above rotation about the origin, and then translating the Euclidean plane by (x_1, y_1). Thus,

$$\begin{aligned} \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} x - x_1 \\ y - y_1 \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \\ &= \begin{pmatrix} x\cos(\phi) - y\sin(\phi) + x_1(1 - \cos(\phi)) + y_1\sin(\phi) \\ x\sin(\phi) + y\cos(\phi) - x_1\sin(\phi) + y_1(1 - \cos(\phi)) \end{pmatrix}. \end{aligned}$$

We may rewrite this as

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos(\phi) & -\sin(\phi) & x_1(1 - \cos(\phi)) + y_1\sin(\phi) \\ \sin(\phi) & \cos(\phi) & -x_1\sin(\phi) + y_1(1 - \cos(\phi)) \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

Thus, a rotation f can also be viewed in terms of matrix multiplication. We call the above 3×3 matrix $R_{\phi,A}$. With the above information, we may now view a combination of translations and rotations in terms of matrix multiplication. For example, if we wish to translate the Euclidean plane by $A = (2, 3)$ and then rotate about the point $B = (1, 1)$ by $\pi/6$ and then translate by $C = (-5, 7)$, each point (x, y) in the Euclidean plane will be moved to (x', y') where

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = T_C R_{\pi/6,B} T_A \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

This is a good place to do some examples and to make up some related homework. Our main goal here is to establish and apply the following result.

Theorem. *Let α and β be real numbers (not necessarily distinct), and let A and B be points (not necessarily distinct). If $\alpha + \beta$ is not an integer multiple of 2π, then there is point C such that $R_{\beta,B}R_{\alpha,A} = R_{\alpha+\beta,C}$. If $\alpha + \beta$ is an integer multiple of 2π, then $R_{\beta,B}R_{\alpha,A}$ is a translation.*

Before demonstrating the theorem it would be a good idea to discuss the analogous result for a composition of 2 translations, the first by (a, b) and the second by (c, d).

Geometrically, it should be clear that the result of such a composition is a translation by $(a+c, b+d)$. Alternatively, one can show by taking the product of matrices that $T_{(a,b)}T_{(c,d)} = T_{(a+c,b+d)}$.

To see why the theorem holds, write $A = (x_1, y_1)$ and $B = (x_2, y_2)$. Then

$$\begin{aligned}
R_{\beta,B}R_{\alpha,A} &= \begin{pmatrix} \cos(\beta) & -\sin(\beta) & x_2(1-\cos(\beta)) + y_2\sin(\beta) \\ \sin(\beta) & \cos(\beta) & -x_2\sin(\beta) + y_2(1-\cos(\beta)) \\ 0 & 0 & 1 \end{pmatrix} \\
&\qquad \times \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) & x_1(1-\cos(\alpha)) + y_1\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) & -x_1\sin(\alpha) + y_1(1-\cos(\alpha)) \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) & u \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) & v \\ 0 & 0 & 1 \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
u &= x_1\cos(\beta)(1-\cos(\alpha)) + y_1\sin(\alpha)\cos(\beta) + x_1\sin(\alpha)\sin(\beta) \\
&\qquad - y_1\sin(\beta)(1-\cos(\alpha)) + x_2(1-\cos(\beta)) + y_2\sin(\beta) \\
&= x_1(1-\cos(\alpha+\beta)) + y_1\sin(\alpha+\beta) \\
&\qquad + (x_2-x_1)(1-\cos(\beta)) + (y_2-y_1)\sin(\beta)
\end{aligned}$$

and

$$\begin{aligned}
v &= x_1\sin(\beta)(1-\cos(\alpha)) + y_1\sin(\alpha)\sin(\beta) - x_1\cos(\alpha)\sin(\beta) \\
&\qquad + y_1\cos(\beta)(1-\cos(\alpha)) - x_2\sin(\beta) + y_2(1-\cos(\beta)) \\
&= -x_1\sin(\alpha+\beta) + y_1(1-\cos(\alpha+\beta)) \\
&\qquad - (x_2-x_1)\sin(\beta) + (y_2-y_1)(1-\cos(\beta)).
\end{aligned}$$

Observe that if $\alpha+\beta$ is an integer multiple of 2π, then the above matrix represents a translation by (u, v) so that the second part of the theorem follows. Suppose now that $\alpha+\beta$ is not an integer multiple of 2π. We will have that there is a C such that $R_{\beta,B}R_{\alpha,A}$ is a rotation at C by the angle $\alpha+\beta$ if we can find a pair (x_3, y_3) such that

$$x_3(1-\cos(\alpha+\beta)) + y_3\sin(\alpha+\beta) = (x_2-x_1)(1-\cos(\beta)) + (y_2-y_1)\sin(\beta)$$

and

$$-x_3\sin(\alpha+\beta) + y_3(1-\cos(\alpha+\beta)) = -(x_2-x_1)\sin(\beta) + (y_2-y_1)(1-\cos(\beta)).$$

We have two equations in the 2 unknowns x_3 and y_3. There is a solution provided that

$$\det\begin{pmatrix} 1-\cos(\alpha+\beta) & \sin(\alpha+\beta) \\ -\sin(\alpha+\beta) & 1-\cos(\alpha+\beta) \end{pmatrix} \neq 0.$$

Observe that one does not need to use anything fancy here; simply solve for x_3 and y_3 above and the equivalent of the determinant being non-zero above follows. We get that C exists provided that

$$2-2\cos(\alpha+\beta) \neq 0.$$

Since we are now only considering $\alpha+\beta$ which are not integer multiples of 2π, the theorem is established.