Math 532: Quiz 2

Name **ANSWERS**

Axiom P1:	There exist at least 4 points no 3 of which are collinear.
Axiom P2:	There exists at least 1 line with exactly $n+1$ (distinct) points on it.
Axiom P3:	Given 2 distinct points, there is exactly 1 line that they both lie on.
Axiom P4:	Given 2 distinct lines, there is at least 1 point on both of them.

In this problem, you are to help provide a proof for the following result done in class:

Result: *If ℓ is a line with exactly $n+1$ points on it in a finite projective plane of order n and A is a point not on ℓ, then there exist exactly $n+1$ lines passing through A.*

You should not use any results that we have already established about finite projective planes. Where I have left room for explanations, be sure to justify your comments using the axioms given above. Note that the quiz includes needed work to be done on the back of this page.

Proof: Let $P_1, P_2, \ldots, P_{n+1}$ be the $n+1$ points on ℓ. In the box below, indicate which axiom is used to get lines $\ell_1, \ell_2, \ldots, \ell_{n+1}$ passing through A and explain what these lines are (i.e., how you are using the axiom).

Axiom P3 implies that, for each $j \in \{1, 2, \ldots, n+1\}$, there is a line ℓ_j passing through A and P_j.

Box 1

In the next box, explain why each of the $n+1$ lines $\ell_1, \ell_2, \ldots, \ell_{n+1}$ is not equal to ℓ.

Since A is on each ℓ_j and A is not on ℓ, we see that $\ell_j \neq \ell$ for each $j \in \{1, 2, \ldots, n+1\}$.

Box 2

In the next box, justify that the $n+1$ lines $\ell_1, \ell_2, \ldots, \ell_{n+1}$ are different. In other words, justify that if i and j are in $\{1, 2, \ldots, n+1\}$ with $i \neq j$, then $\ell_i \neq \ell_j$. You should use one of the axioms and the information from Box 2 above. Clarify where you use them.

Assume $\ell_i = \ell_j$ for i and j in $\{1, 2, \ldots, n+1\}$ with $i \neq j$. Then P_i and P_j are both on ℓ_i. Since $P_i \neq P_j$, Axiom P3 implies that there is exactly one line passing through P_i and P_j. Since P_i and P_j are both on ℓ_i and are both on ℓ, we deduce $\ell_i = \ell$. This contradicts Box 2 above. So our assumption is wrong and the lines ℓ_j are different.

Box 3

(turn over to complete the quiz)

The above justifies that there are at least $n + 1$ different lines passing through A. To finish the proof, we need to show that there are no more lines passing through A. Let ℓ' be a line passing through A. Justify that $\ell' = \ell_j$ for some j in $\{1, 2, \ldots, n + 1\}$. Your justification should refer to more than one of the axioms.

Since A is on ℓ' and A is not on ℓ, we have $\ell' \neq \ell$. By Axiom P4, there is a point that lies on both ℓ' and ℓ. Since the only points on ℓ are $P_1, P_2, \ldots, P_{n+1}$, we deduce that P_j is on ℓ' for some $j \in \{1, 2, \ldots, n + 1\}$. Since P_j is on ℓ and A is not on ℓ, we have $P_j \neq A$. By Axiom P3, there is exactly one line passing through P_j and A. Since both ℓ' and ℓ_j pass through P_j and A, we deduce $\ell' = \ell_j$.

Box 4

The above shows that there are at least $n + 1$ lines passing through A and that there are no more than $n + 1$ lines passing through A. Hence, there are exactly $n + 1$ lines passing through A. This completes the proof of the result. ■