MATH 532, 736I: MODERN GEOMETRY

Name ________________________________

Final Exam (1994)

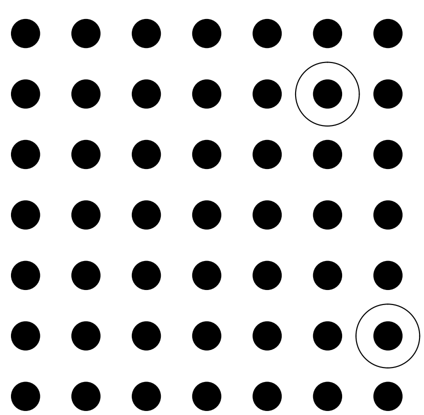
Show All Work

There are 200 total points on the Exam. The first 100 points are related to the first half of the course, and the second 100 points to the second half of the course. The value of the problems is as indicated below.

Problem 1	14 points
Problem 2	10 points
Problem 3	18 points
Problem 4	18 points
Problem 5 (a)	6 points
Problem 5 (b)	8 points
Problem 5 (c)	6 points
Problem 5 (d)	8 points
Problem 5 (e)	12 points
Problem 6	16 points
Problem 7	16 points
Problem 8	18 points
Problem 9	16 points
Problem 10	18 points
Problem 11	16 points

(1) State the axioms for a finite projective plane of order n.

(2) Two points have been circled in the 7×7 array of points below. Using the model for a finite affine plane of order 7 discussed in class, finish circling the points that belong to the same line as the given circled points.



(3) Consider the 41×41 array of points for our model of a finite affine plane of order 41. The lines $y \equiv 20x + 17 \pmod{41}$ and $y \equiv 23x + 19 \pmod{41}$ intersect at some point (a, b). Determine the values of a and b and put them in the indicated boxes below. Both a and b should be among the 42 numbers $0, 1, 2, \ldots, 41$.

$a = \boxed{}$

$b = \boxed{}$

(4) Using only the results about affine planes below and the axioms stated at the end of this test (and referring to them whenever you use them), prove that in an affine plane of order n, each point has exactly $n + 1$ lines passing through it. This is a proof you were to have memorized for class. Note that each result below involves a point and a line. Whenever you use one of the results below, be sure to clarify what point and line you are using with the result.

Result 1: *If ℓ is a line with exactly n points on it and A is a point not on ℓ, then there are exactly $n + 1$ lines passing through A.*

Result 2: *If A is a point with exactly $n + 1$ lines passing through it and ℓ is a line with A not on ℓ, then there are exactly n points on ℓ.*

(Do NOT prove Result 1 or Result 2!!!)

(5) The following problems (parts (a) through (e)) all deal with an axiomatic system consisting of the axioms below.

Axiom 1. There exists at least 1 line with exactly 2 distinct points on it.

Axiom 2. There exists at least 1 line with exactly 3 distinct points on it.

Axiom 3. Given any 2 distinct points, there exists exactly one line passing through the 2 points.

Axiom 4. Given any line ℓ and any point P not on ℓ, there is at least one line through P that does not intersect ℓ.

(a) Explain why the axiomatic system is consistent.

(b) Explain why the axiomatic system is independent.

(c) Explain why the axiomatic system is NOT complete.

(d) Does the principle of duality hold? Justify your answer.

(e) Prove that there is at least one line ℓ such that at least 2 distinct lines ℓ_1 and ℓ_2 do not intersect ℓ.

(6) Let the function $f(x, y)$ be defined as follows. First f rotates each point (x, y) about the point $(0, 0)$ by $\pi/2$, then it takes the result and rotates it about $(1, 0)$ by π, and then it takes that result and rotates it about the point $(0, 1)$ by $\pi/2$. Determine whether f is a translation or a rotation and write it in the form T_B or $R_{\alpha,A}$ giving specific values for B, α, and A if appropriate (i.e., if $f = T_B$, tell me what B is, and if $f = R_{\alpha,A}$, tell me what α and A are).

(7) On the third to the last page is a drawing consisting of 2 triangles ΔABC and $\Delta A'B'C'$ which are perspective from the line passing through P, Q, and R. Suppose we want to use Desargues' Theorem to prove that these triangles are also perspective from a point. Let X be the point of intersection of the lines $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$. To show that $\overleftrightarrow{AA'}$ also passes through X, we can consider two triangles which we know to be perspective from a point. The vertices of these two triangles (not necessarily the edges) and the point they are perspective from are all in the drawing. Determine the two triangles and the point they are perspective from and indicate your answers in the boxes below. (No work necessary.)

Triangles $\boxed{}$ and $\boxed{}$ are perspective from point $\boxed{}$.

(Note: You can complete the boxes to make a sentence which is true, but that will not necessarily insure that you have a correct answer. A correct answer must correspond to establishing that $\overleftrightarrow{AA'}$ passes through X as discussed above.)

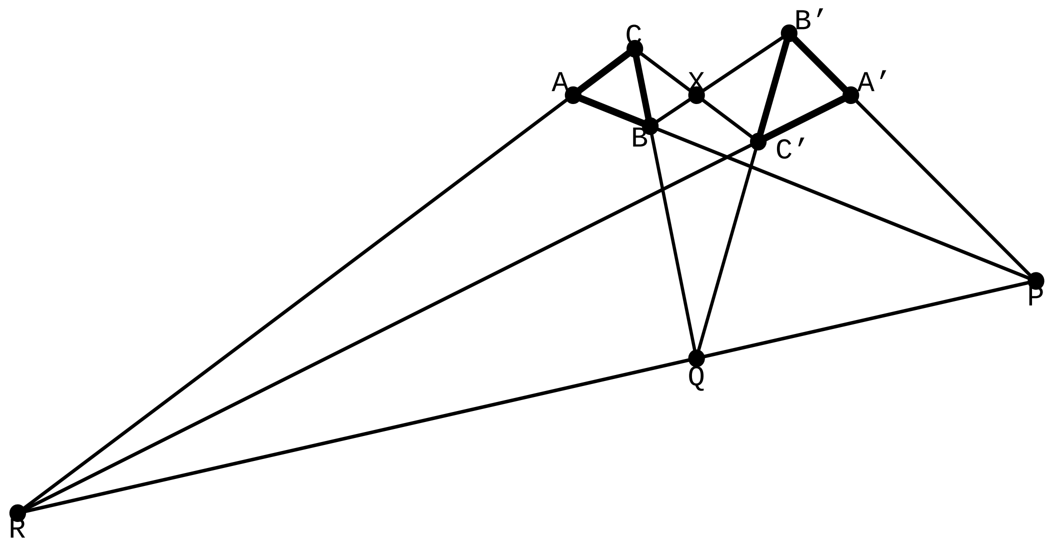
(8) A quadrilateral has 4 vertices A, B, C, and D oriented counter-clockwise (a picture for this problem is on the third to the last page of the test). Suppose that $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are perpendicular and that $\overleftrightarrow{DB}$ and $\overleftrightarrow{DC}$ are perpendicular. Let M be the midpoint of $\overline{BC}$, and let N be the midpoint of $\overline{AD}$. Using vectors, prove that $\overleftrightarrow{MN}$ is perpendicular to $\overleftrightarrow{AD}$.

(9) Let f be a rotation (by some angle about some point). Let $A = (0, 52)$, $B = (0, 25)$, and $C = (39, 0)$. Suppose $f(A) = B$ and $f(C) = (x, 0)$ for some numbr x. Determine with proof the value of x.

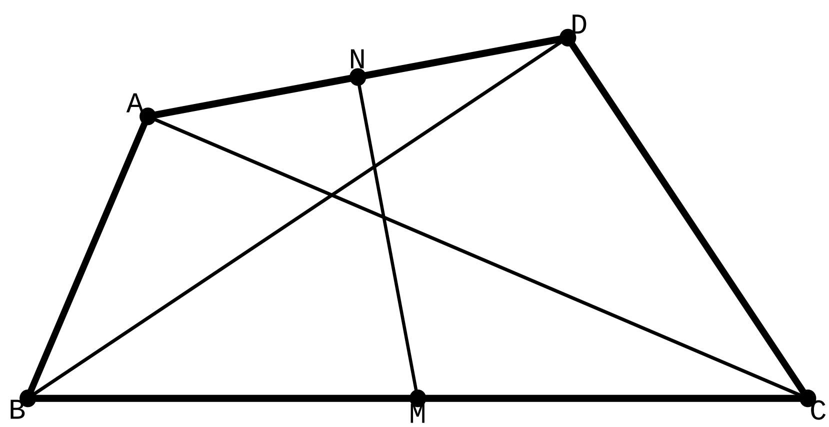
(10) Let $P_1, P_2, \ldots, P_n$ be n (not necessarily distinct) points in the plane. Let A and B be two arbitrary points. Beginning with $A_0 = A$, for $j \in \{1, 2, \ldots, n\}$, define A_j as the point you get by rotating A_{j-1} about P_j by π. Beginning with $B_0 = B$, for $j \in \{1, 2, \ldots, n\}$, define B_j as the point you get by rotating B_{j-1} about P_j by π. Prove that A, B, A_n, and B_n either all lie on a line or they are the vertices of a parallelogram. Your answer should make use of Theorem 1 on the last page of this exam. (Comment: You may want to consider even and odd n separately.)

(11) Explain how the theorem below follows from Desargues' Theorem. The answer should be fairly short but, at the same time, you should be sure to be very precise. A picture demonstrating the statement of the theorem can be found on the second to the last page of this test. (Hint: Think in terms of an infinite projective plane.)

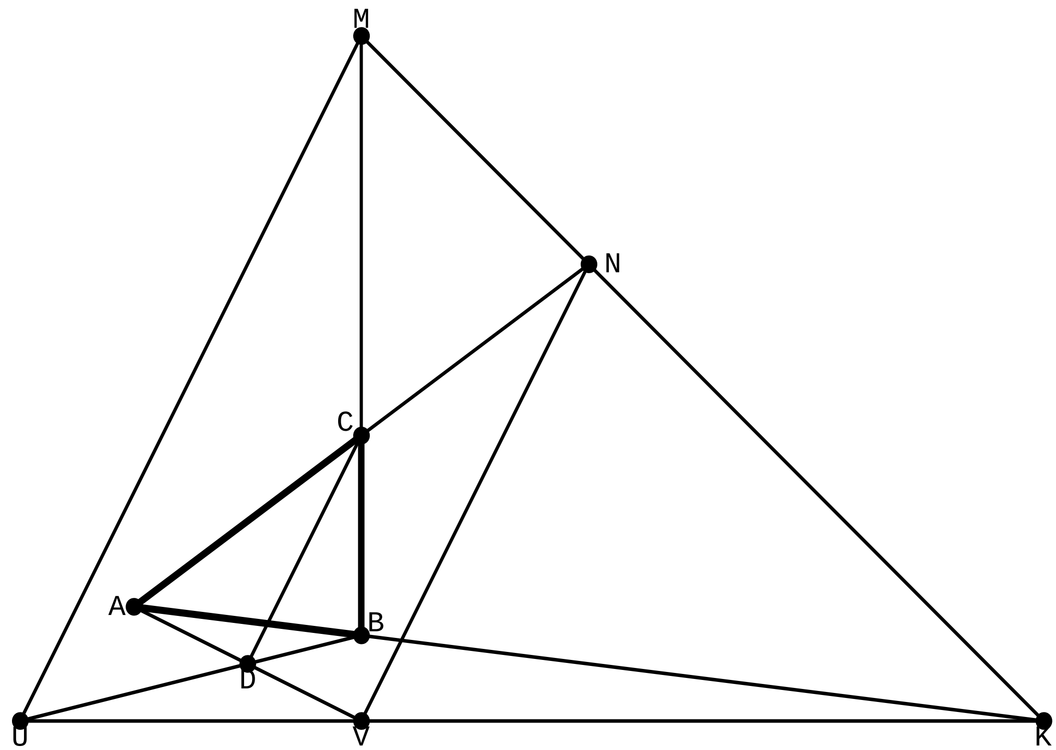
Theorem: *Let ΔABC be an arbitrary triangle, and let U and V be 2 arbitrary distinct points with $U \neq B$ and $V \neq A$. Let D be the intersection of $\overleftrightarrow{AV}$ and $\overleftrightarrow{BU}$. Suppose $D \neq C$. Let ℓ be the line $\overleftrightarrow{CD}$. Suppose that $\overleftrightarrow{AB}$ and $\overleftrightarrow{UV}$ are distinct lines intersecting at a point K, that the line through U parallel to ℓ and the line $\overleftrightarrow{BC}$ are distinct lines intersecting at a point M, and that the line through V parallel to ℓ and the line $\overleftrightarrow{AC}$ are distinct lines intersecting at a point N (with each of K, M, and N points NOT at infinity). Then the points K, M, and N are collinear.*



Problem 7



Problem 8



Problem 11

INFORMATION PAGE

Axiom A1. There exist at least 4 distinct points no 3 of which are collinear.

Axiom A2. There exists at least 1 line with exactly n points on it.

Axiom A3. Given any 2 distinct points, there exists exactly one line passing through the 2 points.

Axiom A4. Given any line ℓ and any point P not on ℓ, there is exactly 1 line through P that does not intersect ℓ.

Theorem 1: *Let α and β be real numbers (not necessarily distinct), and let A and B be points (not necessarily distinct). If $\alpha + \beta$ is not an integer multiple of 2π, then there is point C such that $R_{\beta,B}R_{\alpha,A} = R_{\alpha+\beta,C}$. If $\alpha + \beta$ is an integer multiple of 2π, then $R_{\beta,B}R_{\alpha,A}$ is a translation.*

$$T_{(a,b)} = \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_{\theta,(x_1,y_1)} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & x_1(1-\cos(\theta)) + y_1 \sin(\theta) \\ \sin(\theta) & \cos(\theta) & -x_1 \sin(\theta) + y_1(1-\cos(\theta)) \\ 0 & 0 & 1 \end{pmatrix}$$

$$T_{(a,b)} = R_{\pi,(a/2,b/2)} R_{\pi,(0,0)}$$