# Math 241: Quiz 11

**Show ALL Work** Name **Solutions**

1. Calculate the value of

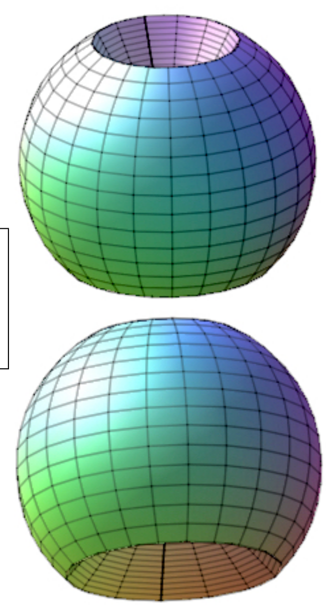
$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{4}\left(x^2+y^2\right)^{3/2}\,dz\,dy\,dx.$$

Answer: $4\pi/5$

**Solution:** We convert the integral to cylindrical coordinates (polar coordinates would also work after first integrating with respect to $z$). The solid described by the limits of integration is half of a cylinder with the base in the $xy$-plane consisting of the semi-circle of radius $1$ centered at the origin and lying on and above the $x$-axis. The top of the half cylinder is bounded by $z = 4$. Since $x^2 + y^2 = r^2$ and $dz\,dy\,dx = r\,dz\,dr\,d\theta$, the given integral is equivalent to

$$\begin{aligned}\int_0^{\pi}\int_0^{1}\int_0^{4}(r^2)^{3/2}r\,dz\,dr\,d\theta &= \int_0^{\pi}\int_0^{1}\int_0^{4} r^4\,dz\,dr\,d\theta \\ &= 4\int_0^{\pi}\int_0^{1} r^4\,dr\,d\theta = \frac{4}{5}\int_0^{\pi}d\theta = \frac{4\pi}{5}. \quad \blacksquare\end{aligned}$$

2. Express the volume of the solid that lies inside the sphere $x^2 + y^2 + z^2 = 4$ and between the two half-cones given by $z = \sqrt{(x^2 + y^2)/3}$ and $z = -\sqrt{x^2 + y^2}$ as a triple integral in spherical coordinates. Do not use inverse trigonometric functions in your answer.



Triple Integral: $$\int_0^{2\pi}\int_{\pi/3}^{3\pi/4}\int_0^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

**Solution:** Note the picture is not an accurate drawing (the top cone should cut out more of the sphere than the bottom cone). The sphere has radius $2$ and center $(0, 0, 0)$. The solid goes completely around the $z$-axis so that $\theta$ ranges from $0$ to $2\pi$. The angle $\phi$ goes from the angle from the positive $z$-axis of the top cone $z = \sqrt{(x^2 + y^2)/3}$ to the angle from the positive $z$-axis of the bottom cone $z = -\sqrt{x^2 + y^2}$. To calculate these angles, we convert the equations of the cones to spherical coordinates. For $z = \sqrt{(x^2 + y^2)/3}$, we have

$$\rho \cos \phi = \frac{r}{\sqrt{3}} = \frac{\rho \sin \phi}{\sqrt{3}}.$$

Rewriting this, we obtain $\tan \phi = \sqrt{3}$. Since $0 \leq \phi \leq \pi$, the top cone is an angle $\pi/3$ from the positive $z$-axis. Converting the bottom cone $z = -\sqrt{x^2 + y^2}$ to spherical coordinates, we obtain $\rho \cos \phi = -r = -\rho \sin \phi$ so that $\tan \phi = -1$. Since $0 \leq \phi \leq \pi$, the bottom cone is an angle $3\pi/4$ from the positive $z$-axis. Thus, $\phi$ goes from $\pi/3$ to $3\pi/4$. Finally, since the sphere has radius $2$, the value of $\rho$ ranges from $0$ to $2$. ■