Math 241: Quiz 10

Show ALL Work Name **Solution**

1. Read the instructions carefully. The function

$$f(x,y) = 2x^2 + x - xy^2 - 2y^2$$

has an absolute maximum value and an absolute minimum value on the set

$$D = \{(x,y) \mid x^2 + y^2 \le 4\}.$$

Find the *points* (x,y) on D (that is, the points inside or on the circle $x^2 + y^2 = 4$) where the absolute maximum and absolute minimum occur. There may be more than one point for one or both answers. You will be graded on your work, so be sure to show appropriate work.

The absolute maximum value occurs at $\boxed{(2,0)}$.

The absolute minimum value occurs at $\boxed{(1/3, \pm\sqrt{35}/3)}$.

Detailed Solution. We want

$$f_x = 4x + 1 - y^2 = 0 \quad \text{AND} \quad f_y = -2xy - 4y = -2y(x+2) = 0.$$

Given $-2y(x+2) = 0$, we know that $y = 0$ OR $x = -2$. Either of these imply $f_y = 0$. We still need $f_x = 0$. If $x = -2$, then $f_x = 4x + 1 - y^2 = 0$ implies $y^2 = -7$ which is impossible. So $x \ne -2$. If $y = 0$, then $f_x = 4x + 1 - y^2 = 0$ implies $x = -1/4$. Thus, we get the one point $(-1/4, 0)$ inside D (not on the boundary of D) that we need to consider. We note that

$$f(-1/4, 0) = 2(-1/4)^2 + (-1/4) = 1/8 - 1/4 = -1/8.$$

On the boundary of D, we have the circle $x^2 + y^2 = 4$. So if (x,y) is on this circle, then $y^2 = 4 - x^2$ and

$$f(x,y) = 2x^2 + x - xy^2 - 2y^2 = 2x^2 + x - x(4 - x^2) - 2(4 - x^2).$$

We set $g(x)$ to be the right side above and simplify to get $g(x) = x^3 + 4x^2 - 3x - 8$. Thus, if (x,y) is on the circle $x^2 + y^2 = 4$, then $f(x,y) = g(x)$ and $-2 \le x \le 2$. We are left then with maximizing and minimizing the function $g(x)$ on the interval $-2 \le x \le 2$. Setting

$$g'(x) = 3x^2 + 8x - 3 = (3x - 1)(x + 3) = 0$$

and noting the -3 does not satisfy $-2 \leq x \leq 2$, we see that the values of x we need to consider with $-2 \leq x \leq 2$ are $x = -2, 1/3$ and 2 (where we have included the endpoints -2 and 2 as needed). Recalling that $f(-1/4, 0) = -1/8$ and using that

$$g(-2) = -8 + 16 + 6 - 8 = 6, \quad g(1/3) = \frac{1}{27} + \frac{4}{27} - 1 - 8 < -8,$$

and

$$g(2) = 8 + 16 - 6 - 8 = 10,$$

we see that the maximum value occurs when (x, y) is on the boundary $x^2 + y^2 = 4$ and $x = 2$. Also, the minimum occurs when (x, y) is on the boundary $x^2 + y^2 = 4$ and $x = 1/3$. If $x^2 + y^2 = 4$ and $x = 2$, then $y = 0$. If $x^2 + y^2 = 4$ and $x = 1/3$, then $y = \pm\sqrt{4 - (1/9)} = \pm\sqrt{35}/3$. The given answers follow.