

Some Solutions to the Final Exam from Spring 2001

Part I, Problem 5: The extrema must occur at points (x, y) satisfying

$$3x^2y = \lambda 12^3, \quad x^3 = \lambda 4y^3, \quad \text{and} \quad 3x^4 + y^4 = 1,$$

for some λ . When you do Lagrange multiplier problems, don't forget the constraint equation (the last equation above). If $x = 0$, then the first two equations can hold by taking $\lambda = 0$. In order for the third equation also to hold, we need $y = \pm 1$. So we $(0, \pm 1)$ are two points to consider for extrema. Next suppose that $x \neq 0$. Then we divide by $3x^2$ in the first equation to get $y = 4\lambda x$. Putting this into the second equation and dividing by x^3 gives $1 = (4\lambda)^4$. So $\lambda = \pm 1/4$. Since $y = 4\lambda x$, we get $y = \pm x$. Putting this into the third equation gives $4x^4 = 1$ so that $x = \pm 1/\sqrt[4]{2}$ and $y = \pm 1/\sqrt[4]{2}$. We are wanting the extrema for $f(x, y) = x^3y$. The points (x, y) to consider are

$$(0, \pm 1), (1/\sqrt[4]{2}, \pm 1/\sqrt[4]{2}), (-1/\sqrt[4]{2}, \pm 1/\sqrt[4]{2}).$$

The maximum value is $f(1/\sqrt[4]{2}, 1/\sqrt[4]{2}) = f(-1/\sqrt[4]{2}, -1/\sqrt[4]{2}) = 1/4$ and the minimum value is $f(1/\sqrt[4]{2}, -1/\sqrt[4]{2}) = f(-1/\sqrt[4]{2}, 1/\sqrt[4]{2}) = -1/4$.

Part II, Problem 1: Let S be the square region enclosed by the C , and let

$$f(x, y) = yx + 3x^2 \sin y + 3y \quad \text{and} \quad g(x, y) = x^3 \cos y - y^3 + 3x.$$

Then Green's Theorem gives us that

$$\begin{aligned} \int_C f(x, y) dx + g(x, y) dy &= \iint_S \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA = \iint_S ((3x^2 \cos y + 3) - (x + 3x^2 \cos y + 3)) dA \\ &= \int_0^1 \int_0^1 (-x) dy dx = \int_0^1 (-x) dx = -\frac{x^2}{2} \Big|_0^1 = -\frac{1}{2}. \end{aligned}$$

Part II, Problem 2: (a) Note that D is the *square* of the distance between a point on ℓ and a point on ℓ' . Since a point on ℓ can be represented as $(1-t, -1-t, 2)$ and a point on ℓ' can be represented at $(-1+s, 3+s, -1+3s)$, the distance formula gives

$$\begin{aligned} D &= ((1-t) - (-1+s))^2 + ((-1-t) - (3+s))^2 + (2 - (-1+3s))^2 \\ &= (2-t-s)^2 + (-4-t-s)^2 + (3-3s)^2. \end{aligned}$$

This last expression for D (or the first one) is sufficient for an answer to this problem.

(b) We want the minimum of D , so we set its partial derivatives equal to 0. This gives

$$-2(2-t-s) - 2(-4-t-s) = 0 \quad \text{and} \quad -2(2-t-s) - 2(-4-t-s) - 6(3-3s) = 0.$$

Subtracting, we see that $6(3 - 3s) = 0$ so that $s = 1$. Putting this in the first (or second) equation gives after simplifying that $t = -2$. Since a minimum distance between the lines must exist, it occurs where $t = -2$ and $s = 1$.

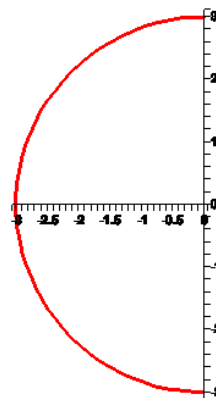
(c) Recall that the point on line ℓ is $(1 - t, -1 - t, 2)$ and the point on ℓ' is $(-1 + s, 3 + s, -1 + 3s)$. At the minimum for D , we have $t = -2$ and $s = 1$, so the points are $(3, 1, 2)$ and $(0, 4, 2)$.

Part II, Problem 3: (a) The integral is over the half-circle of radius 3 shown to the right. Switching to polar coordinates, the integral becomes

$$\int_{\pi/2}^{3\pi/2} \int_0^3 \sqrt{r^2 + 16} r dr d\theta.$$

Evaluating by using the substitution $u = r^2 + 16$ gives that this integral is the same as

$$\int_{\pi/2}^{3\pi/2} \frac{1}{3} (r^2 + 16)^{3/2} \Big|_0^3 d\theta = \int_{\pi/2}^{3\pi/2} \frac{1}{3} (5^3 - 4^3) \Big|_0^3 d\theta = \frac{61\pi}{3}.$$

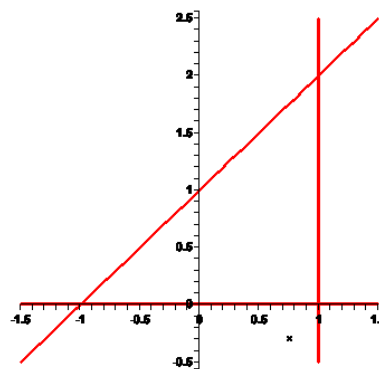


(b) The integral is over the triangle depicted to the right. Switching the order of integration results in the integral

$$\int_0^2 \int_{y-1}^1 (4y - y^2)^{3/2} dx dy = \int_0^2 (2 - y)(4y - y^2)^{3/2} dy.$$

Evaluating this last integral by substituting $u = 4y - y^2$ results in

$$\frac{1}{5} (4y - y^2)^{5/2} \Big|_0^2 = \frac{32}{5}.$$



Part II, Problem 4: (a) Suppose $ax + by + cz = d$ is the plane \mathcal{P}' . Since \mathcal{P}' intersects \mathcal{P} at a 60° angle, there is either a 60° angle or a 120° angle between a normal to the plane \mathcal{P}' and a normal to the plane \mathcal{P} . In other words, the angle between $\langle a, b, c \rangle$ (which is normal to \mathcal{P}') and $\langle 1, -2, -1 \rangle$ (which is normal to \mathcal{P}) is either 60° or 120° . This gives

$$\pm \frac{1}{2} = \frac{\langle a, b, c \rangle \cdot \langle 1, -2, -1 \rangle}{\|\langle a, b, c \rangle\| \|\langle 1, -2, -1 \rangle\|} = \frac{a - 2b - c}{\sqrt{a^2 + b^2 + c^2} \sqrt{6}},$$

which is equivalent to what was to be shown. The exact same argument applies above if $ax + by + cz = d$ is the plane \mathcal{P}'' instead of the plane \mathcal{P}' .

(b) Putting $a^2 + b^2 + c^2 = 6$ into the formula for part (a) gives

$$a - 2b - c = \pm \frac{\sqrt{6}}{2} \sqrt{6} = \pm \frac{6}{2} = \pm 3.$$

This is all that was asked for.

(c) The points Q and R are on both the planes \mathcal{P}' and \mathcal{P}'' . This means that \overrightarrow{QR} is parallel to these planes. Since $\langle a, b, c \rangle$ is a vector that is perpendicular to one of these planes, we obtain that \overrightarrow{QR} and $\langle a, b, c \rangle$ are perpendicular. This implies their dot product is 0. Since $Q = (3, -2, 4)$ and $R = (4, -1, 3)$, we have $\overrightarrow{QR} = \langle 1, 1, -1 \rangle$. Hence, $\langle 1, 1, -1 \rangle \cdot \langle a, b, c \rangle = 0$, which gives $a + b - c = 0$.

(d) Subtracting $a - 2b - c = 3$ from $a + b - c = 0$ gives $3b = -3$ so that $b = -1$. The equations become $a - c = 1$ and $a^2 + c^2 = 5$. The first of these implies $a = c + 1$. Plugging this into the other equation gives $(c + 1)^2 + c^2 = 5$ which, after a little work, is equivalent to $c^2 + c - 2 = 0$. Since $c^2 + c - 2 = (c - 1)(c + 2)$, we deduce that either $c = 1$ or $c = -2$. Recalling that $a = c + 1$ and $b = -1$, we deduce that either $(a, b, c) = (2, -1, 1)$ or $(a, b, c) = (-1, -1, -2)$.

(e) There are two planes passing through Q and R that make a 60° angle with \mathcal{P} . Part (d) implies that they are of the form $2x - y + z = d_1$ and $-x - y - 2z = d_2$ for some numbers d_1 and d_2 . Since $Q = (3, -2, 4)$ is on each plane, we deduce $d_1 = 12$ and $d_2 = -9$. Hence, the planes \mathcal{P}' and \mathcal{P}'' can be written in the form $2x - y + z = 12$ and $x + y + 2z = 9$.