Some Solutions to the Final Exam from Spring 2001

Part I, Problem 5: The extrema must occur at points (x, y) satisfying

$$3x^2y = \lambda 12x^3, \quad x^3 = \lambda 4y^3, \quad \text{and} \quad 3x^4 + y^4 = 1,$$

for some λ. When you do Lagrange multiplier problems, don't forget the constraint equation (the last equation above). If $x = 0$, then the first two equations can hold by taking $\lambda = 0$. In order for the third equation also to hold, we need $y = \pm 1$. So $(0, \pm 1)$ are two points to consider for extrema. Next suppose that $x \neq 0$. Then we divide by $3x^2$ in the first equation to get $y = 4\lambda x$. Putting this into the second equation and dividing by x^3 gives $1 = (4\lambda)^4$. So $\lambda = \pm 1/4$. Since $y = 4\lambda x$, we get $y = \pm x$. Putting this into the third equation gives $4x^4 = 1$ so that $x = \pm 1/\sqrt{2}$ and $y = \pm 1/\sqrt{2}$. We are wanting the extrema for $f(x, y) = x^3 y$. The points (x, y) to consider are

$$(0, \pm 1),\ (1/\sqrt{2}, \pm 1/\sqrt{2}),\ (-1/\sqrt{2}, \pm 1/\sqrt{2}).$$

The maximum value is $f(1/\sqrt{2}, 1/\sqrt{2}) = f(-1/\sqrt{2}, -1/\sqrt{2}) = 1/4$ and the minimum value is $f(1/\sqrt{2}, -1/\sqrt{2}) = f(-1/\sqrt{2}, 1/\sqrt{2}) = -1/4$.

Part II, Problem 1: Let S be the square region enclosed by the C, and let

$$f(x, y) = yx + 3x^2 \sin y + 3y \quad \text{and} \quad g(x, y) = x^3 \cos y - y^3 + 3x.$$

Then Green's Theorem gives us that

$$\int_C f(x, y)\, dx + g(x, y)\, dy = \iint_S \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dA = \iint_S \big((3x^2 \cos y + 3) - (x + 3x^2 \cos y + 3) \big)\, dA$$

$$= \int_0^1 \int_0^1 (-x)\, dy\, dx = \int_0^1 (-x)\, dx = -\frac{x^2}{2}\bigg|_0^1 = -\frac{1}{2}.$$

Part II, Problem 2: **(a)** Note that D is the *square* of the distance between a point on ℓ and a point on ℓ'. Since a point on ℓ can be represented as $(1-t, -1-t, 2)$ and a point on ℓ' can be represented at $(-1+s, 3+s, -1+3s)$, the distance formula gives

$$\begin{aligned} D &= \big((1-t) - (-1+s)\big)^2 + \big((-1-t) - (3+s)\big)^2 + \big(2 - (-1+3s)\big)^2 \\ &= (2-t-s)^2 + (-4-t-s)^2 + (3-3s)^2. \end{aligned}$$

This last expression for D (or the first one) is sufficient for an answer to this problem.

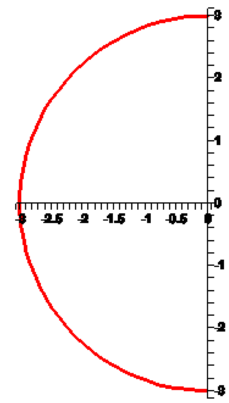
(b) We want the minimum of D, so we set its partial derivatives equal to 0. This gives

$$-2(2-t-s) - 2(-4-t-s) = 0 \ \text{ and } \ -2(2-t-s) - 2(-4-t-s) - 6(3-3s) = 0.$$

Subtracting, we see that $6(3-3s)=0$ so that $s=1$. Putting this in the first (or second) equation gives after simplifying that $t=-2$. Since a minimum distance between the lines must exist, it occurs where $t=-2$ and $s=1$.

(c) Recall that the point on line ℓ is $(1-t,-1-t,2)$ and the point on ℓ' is $(-1+s,3+s,-1+3s)$. At the minimum for D, we have $t=-2$ and $s=1$, so the points are $(3,1,2)$ and $(0,4,2)$.

Part II, Problem 3: **(a)** The integral is over the half-circle of radius 3 shown to the right. Switching to polar coordinates, the integral becomes

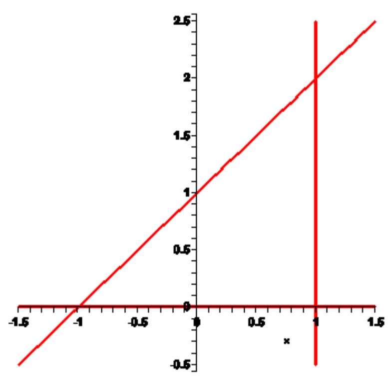


$$\int_{\pi/2}^{3\pi/2}\int_0^3 \sqrt{r^2+16}\, r\, dr\, d\theta.$$

Evaluating by using the substitution $u=r^2+16$ gives that this integral is the same as

$$\int_{\pi/2}^{3\pi/2} \frac{1}{3}(r^2+16)^{3/2}\Big|_0^3 d\theta = \int_{\pi/2}^{3\pi/2} \frac{1}{3}(5^3-4^3)\Big|_0^3 d\theta = \frac{61\pi}{3}.$$

(b) The integral is over the triangle depicted to the right. Switching the order of integration results in the integral



$$\int_0^2\int_{y-1}^1 (4y-y^2)^{3/2}\, dx\, dy = \int_0^2 (2-y)(4y-y^2)^{3/2}\, dy.$$

Evaluating this last integral by substituting $u=4y-y^2$ results in

$$\frac{1}{5}(4y-y^2)^{5/2}\Big|_0^2 = \frac{32}{5}.$$

Part II, Problem 4: **(a)** Suppose $ax+by+cz=d$ is the plane $\mathcal{P}'$. Since $\mathcal{P}'$ intersects $\mathcal{P}$ at a $60°$ angle, there is either a $60°$ angle or a $120°$ angle between a normal to the plane $\mathcal{P}'$ and a normal to the plane $\mathcal{P}$. In other words, the angle between $\langle a,b,c\rangle$ (which is normal to $\mathcal{P}'$) and $\langle 1,-2,-1\rangle$ (which is normal to $\mathcal{P}$) is either $60°$ or $120°$. This gives

$$\pm\frac{1}{2} = \frac{\langle a,b,c\rangle\cdot\langle 1,-2,-1\rangle}{\|\langle a,b,c\rangle\|\,\|\langle 1,-2,-1\rangle\|} = \frac{a-2b-c}{\sqrt{a^2+b^2+c^2}\,\sqrt{6}},$$

which is equivalent to what was to be shown. The exact same argument applies above if $ax+by+cz=d$ is the plane $\mathcal{P}''$ instead of the plane $\mathcal{P}'$.

(b) Putting $a^2 + b^2 + c^2 = 6$ into the formula for part (a) gives

$$a - 2b - c = \pm\frac{\sqrt{6}}{2}\sqrt{6} = \pm\frac{6}{2} = \pm 3.$$

This is all that was asked for.

(c) The points Q and R are on both the planes $\mathcal{P}'$ and $\mathcal{P}''$. This means that $\overrightarrow{QR}$ is parallel to these planes. Since $\langle a, b, c\rangle$ is a vector that is perpendicular to one of these planes, we obtain that $\overrightarrow{QR}$ and $\langle a, b, c\rangle$ are perpendicular. This implies their dot product is 0. Since $Q = (3, -2, 4)$ and $R = (4, -1, 3)$, we have $\overrightarrow{QR} = \langle 1, 1, -1\rangle$. Hence, $\langle 1, 1, -1\rangle \cdot \langle a, b, c\rangle = 0$, which gives $a + b - c = 0$.

(d) Subtracting $a - 2b - c = 3$ from $a + b - c = 0$ gives $3b = -3$ so that $b = -1$. The equations become $a - c = 1$ and $a^2 + c^2 = 5$. The first of these implies $a = c + 1$. Plugging this into the other equation gives $(c + 1)^2 + c^2 = 5$ which, after a little work, is equivalent to $c^2 + c - 2 = 0$. Since $c^2 + c - 2 = (c - 1)(c + 2)$, we deduce that either $c = 1$ or $c = -2$. Recalling that $a = c + 1$ and $b = -1$, we deduce that either $(a, b, c) = (2, -1, 1)$ or $(a, b, c) = (-1, -1, -2)$.

(e) There are two planes passing through Q and R that make a $60°$ angle with $\mathcal{P}$. Part (d) implies that they are of the form $2x - y + z = d_1$ and $-x - y - 2z = d_2$ for some numbers d_1 and d_2. Since $Q = (3, -2, 4)$ is on each plane, we deduce $d_1 = 12$ and $d_2 = -9$. Hence, the planes $\mathcal{P}'$ and $\mathcal{P}''$ can be written in the form $2x - y + z = 12$ and $x + y + 2z = 9$.