SOLUTIONS TO THE USC MATHEMATICS CONTEST 1996

(1) (b) One checks each of the five given possible answers. Since $2^3 - 8 = 0$, the answer is 2.

(2) (e) One checks which of the choices is the same as $a*(b*a) = a*(b+2a) = a+2(b+2a) = a + 2b + 4a = 5a + 2b$. Since $(5a) * b = 5a + 2b$, the answer is (e).

(3) (a) Observe that

$$\log_5(\tan x) = \log_5\left(\frac{\sin x}{\cos x}\right) = \log_5(\sin x) - \log_5(\cos x).$$

It follows that $\log_5(\tan x) - \log_5(\sin x) + \log_5(\cos x) = 0$ for $x = 20°$ (and whenever $0 < x < 90°$).

(4) (e) Since $\sin x + \cos x = 1/2$, we get

$$\frac{1}{4} = (\sin x + \cos x)^2 = \sin^2 x + 2\sin x\cos x + \cos^2 x = 1 + 2\sin x\cos x.$$

It follows that $\sin x\cos x = -3/8$. Thus,

$$\sin^3 x + \cos^3 x = (\sin x + \cos x)(\sin^2 x - \sin x\cos x + \cos^2 x) = \frac{1}{2}\left(1 + \frac{3}{8}\right) = \frac{11}{16}.$$

There are a variety of other ways to do this problem. Can you think of other approaches?

(5) (d) Let α be the measure of $\angle ADC$. Then since ΔADC is isosceles, we obtain that the measures of $\angle CAD$ and $\angle DCA$ are α and $180° - 2\alpha$, respectively. It follows that $\angle ACB$ has measure 2α. Since ΔACB is isosceles, we obtain that the measures of $\angle CBA$ and $\angle BAC$ are 2α and $180° - 4\alpha$, respectively. Since ΔADB is isosceles, we deduce that the measures of $\angle CBA$ and $\angle BAD$ are the same. This gives that $2\alpha = (180° - 4\alpha) + \alpha$ which implies $5\alpha = 180°$. Therefore, $\alpha = 36°$.

(6) (b) The choices are a little easier to compare written this way:

(a) $(10^4)^{100} = 10^{400}$ (b) $(2^{10})^{1000} > (10^3)^{1000} = 10^{3000}$ (c) $(10^3)^{1000} = 10^{3000}$

(d) $(5^4)^{1000} < (10^3)^{1000} = 10^{3000}$ (e) $(3^2)^{1000} < 10^{1000}$

The answer is clear.

(7) (d) Since

$$(x^2 + ax + 1)(x^4 + bx^3 + cx^2 + dx + 1) = x^6 + (a+b)x^5 + \cdots + 1,$$

we see that $a + b$ is the coefficient of x^5, which is 4.

(8) (e) We use the change of base formula $\log_b a = \log_c a / \log_c b$ to rewrite $f(n)$ in terms of logarithms to the base 2. Thus,

$$f(n) = \log_2 3 \times \frac{\log_2 4}{\log_2 3} \times \frac{\log_2 5}{\log_2 4} \times \cdots \times \frac{\log_2 n}{\log_2 (n-1)} = \log_2 n$$

(as the other logarithm expressions cancel in the product). Therefore,

$$\sum_{k=2}^{10} f(2^k) = \sum_{k=2}^{10} \log_2(2^k) = \sum_{k=2}^{10} k = 2 + 3 + 4 + \cdots + 10 = 54.$$

(9) (a) If there are n people at the party, then each of the n people will shake hands with $n-1$ other people. Since each handshake accounts for a handshake that each of two people have made (i.e., since there are two people involved with each handshake), the total number of handshakes the n people will make is $n(n-1)/2$. (Alternatively, the number of handshakes is the number of ways of choosing 2 people from among n people, which is $\binom{n}{2} = n(n-1)/2$.) Hence, we have $n(n-1)/2 = 66 = 12 \times 11/2$, so $n = 12$.

(10) (c) If the Lion is telling the truth, the day of the week must be Thursday. If he is lying, then the day of the week must be Monday. So the day of the week must be either Thursday or Monday. If the Unicorn is telling the truth, the day of the week must be Sunday. If he is lying, then the day of the week must be Thursday. The day of the week cannot be Sunday (since we have already said that it must be Thursday or Monday). Therefore, it must be Thursday.

(11) (a) Let $f(x) = x^3 + 8x^2 + 12x - 385$. Since

$$f(6) = 6^3 + 8 \times 6^2 + 12 \times 6 - 385 = 6^2(6 + 8 + 2) - 385 = 36 \times 16 - 385 > 0$$

and $f(0) = -385 < 0$, there must be some $t \in (0, 6)$ for which $f(t) = 0$. Therefore, $f(x)$ has a positive real root between 0 and 6. Descartes' Rule of Signs implies that there is only one positive real root, so the answer must be (a). Or one can observe that $f(5) = 0$ to obtain the answer.

(12) (d) A pattern quickly develops. We have

$$s(1996) = 25, \quad s^2(1996) = s(25) = 7, \quad s^3(1996) = s(s^2(1996)) = s(7) = 7, \ldots.$$

In other words, $s^k(1996) = 7$ for all $k \geq 2$. Thus, $s^{1996}(1996) = 7$.

(13) (d) Since

$$\begin{aligned} h(x) = \frac{f(x)}{g(x)} &= \frac{(x^5 - 1)(x^3 + 1)}{(x^2 - 1)(x^2 - x + 1)} \\ &= \frac{(x-1)(x^4 + x^3 + x^2 + x + 1)(x^3 + 1)}{(x-1)(x+1)(x^2 - x + 1)} \end{aligned}$$

$$= \frac{(x^4+x^3+x^2+x+1)(x^3+1)}{(x+1)(x^2-x+1)},$$

it follows that

$$h(1) = \frac{5\times 2}{2\times 1} = 5.$$

(Observe that $(x+1)(x^2-x+1) = x^3+1$, so $h(x) = x^4+x^3+x^2+x+1$. Computing $h(x)$ is not necessary, but it is certainly a reasonable thing to do.)

(14) (b) One checks directly that the following list of scores up to 14 is the complete list of obtainable scores up to that point: 3, 6, 7, 9, 10, 12, 13, 14. Now, we have 3 consecutive scores, namely 12, 13, and 14, which are obtainable, and this implies every score > 14 is obtainable. To see this, observe that if n is an obtainable score, then so is $n+3$ (simply add another field goal to whatever it took to get n points); hence, $15 = 12+3$, $16 = 13+3$, and $17 = 14+3$ are all obtainable and so are $18 = 15+3$, $19 = 16+3$, $20 = 17+3$, and so on. Therefore, the positive integral scores which are not obtainable are 1, 2, 4, 5, 8, and 11. Thus, the answer is 6.

(15) (e) Let n be the number of problems on the test. Then Dave took $6n$ minutes to complete the test and Michael took 2 hours and n minutes (i.e., $120+n$ minutes) to complete the test. Therefore, $6n = 120+n$ which implies that $n = 24$. Since the amount of time to complete the test was the same for Dave and Michael, it took Dave 2 hours and 24 minutes to answer all the problems.

(16) (c) Let A be the common value of $f(1)$, $f(2)$, $f(3)$, $f(4)$, and $f(5)$. Then $f(x)-A$ has the roots 1, 2, 3, 4, and 5. Since $f(x)-A = x^5+ax^4+bx^3+cx^2+dx+(e-A)$, $f(x)-A$ is a polynomial of degree 5 and, hence, the five numbers 1, 2, 3, 4, and 5 must be the five roots of $f(x)$. It follows that the sum of the roots of $f(x)$ is $1+2+3+4+5 = 15$. Since in general the sum of the n roots of a polynomial $a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$ of degree n is $-a_{n-1}/a_n$, we get that the sum of the roots of $f(x)-A$ is $-a$. Thus, $-a = 15$ so that $a = -15$. (Comment: When we say that in general the sum of the n roots above is $-a_{n-1}/a_n$, we mean that the polynomial may have "multiple" roots and we are to count the roots with multiplicity. For example, $(x-1)^2(x-2)^3 = x^5-8x^4+\cdots$ where 8 represents the sum of the roots counting 1 twice and 2 three times.)

(17) (e) This is a direct application of the Law of Cosines with two sides of a triangle having lengths 10 inches and 16 inches and the angle between them being 60°. If x denotes the distance between the tips of the hands in inches, then

$$x^2 = 10^2+16^2-2\times 10\times 16\times\cos(60^\circ) = 100+256-160 = 196.$$

Hence, $x = 14$.

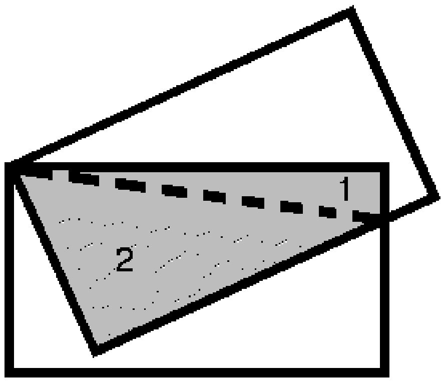
(18) (b) The product of the numbers in all three rows is simply the product of the numbers $1, 2, 4, 8, \ldots, 256$ or, equivalently, of the numbers $2^0, 2^1, 2^2, 2^3, \ldots, 2^8$. The product is

$$2^{0+1+2+\cdots+8} = 2^{36}.$$

This is the product of the numbers in all three rows, so the product of the numbers in one row is $(2^{36})^{1/3} = 2^{12} = 4096$ (since each row has the same product). (Comment: This is a variation of a Magic Square problem. Observe that we only used that the product of the numbers in each row is the same and not the added information about the columns and the diagonals.)

(19) (a) Since (ii) holds for every real number x, it will remain valid if we replace x with $-x$. Therefore, $f(1-x) = 3+f(-x)$. Now, from (i), $11 = f(x)+f(1-x) = f(x)+3+f(-x)$ so that $f(x) + f(-x) = 11 - 3 = 8$. (Observe that $f(x) = 3x + 4$ satisfies the conditions in the problem.)

(20) (d) The area asked for in the problem is the area of the lower rectangle minus the shaded area to the right. The area of this region can be computed by dividing it into two triangles numbered **1** and **2**. These are both right triangles. Triangle **1** has legs of length 1 and 12 so its area is $(1/2) \times 1 \times 12 = 6$. The triangles share a common hypotenuse of length $\sqrt{1^2 + 12^2} = \sqrt{145}$. The legs of Triangle **2** are therefore of length 8 and $\sqrt{145 - 8^2} = \sqrt{81} = 9$. This triangle has area $(1/2) \times 8 \times 9 = 36$. The area of the lower rectangle (or either rectangle) is $8 \times 12 = 96$. Therefore, the answer is $96 - 6 - 36 = 54$.



(21) (d) If there are n socks, r of which are red, in the drawer, then the probability that two chosen together at random are red is

$$\frac{r}{n} \times \frac{r-1}{n-1} = \frac{5}{14}.$$

Therefore, $r(r-1) = 5k$ and $n(n-1) = 14k$ for some positive integer k. Since 14 must divide $n(n-1)$, the only choices given which need be considered are $n = 7$ and $n = 8$. If $n = 7$, then $n(n-1) = 14k$ implies $k = 3$ but then $r(r-1) = 5k = 15$ is an impossibility. If $n = 8$, we get $k = 4$ and $r = 5$, a valid possiblity. Therefore, the answer is $n = 8$.

(22) (c) Count the number of pairs (p, ℓ) where p is one of the n points and ℓ is a line segment joining p to one of the other $n-1$ points. Each point is joined to exactly 3 other points by line segments so that the total number of pairs is $3n$ (three line segments for each of the n points). On the other hand, if there are m total line segments, then each line segment joins exactly two points so that the total number of pairs (p, ℓ) is $2m$ (two points for each of the m line segments). Thus, we must have $3n = 2m$. But this means that n must be even. The only even choice given is 18, so this must be the answer.

Is $n = 18$ really possible or is there some other argument that shows n must be divisible by 23? One can justify that in fact if n is even, then one can join each of any n points to 3 of the remaining $n-1$ points by line segments. To see this, let $n = 2k$, and let the points be $p_1, \ldots, p_n$. For each j, join p_j by line segments to p_{j-1}, p_{j+1}, and p_{j+k} where subscripts are to be interpreted modulo n. Thus, for example, if $n = 18$, then the point p_{12} is joined to p_{11}, p_{13}, and p_3 (since $12 + 9 \equiv 3 \pmod{18}$) and the point p_1 is joined to p_{18} (since $1 - 1 \equiv 18 \pmod{18}$), p_2, and p_{10}. You should convince yourself that this construction for n even joins each point to exactly 3 others.

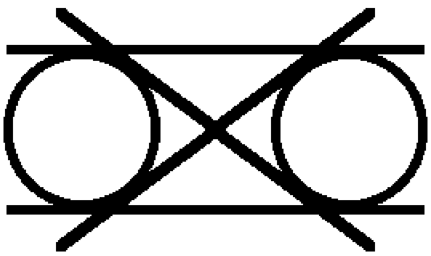
(23) (a) Since minimal investments must be made for 2, 2, 3, and 4 thousand dollars into the four mutual funds, this leaves $20 - 2 - 2 - 3 - 4 = 9$ thousand dollars to invest as one pleases. Thus, we want to determine the number of ways of dividing up 9 thousand dollars among 4 different mutual funds. Consider 12 different boxes aligned as shown and check three of them:

✓				✓					✓		
1	2	3	4	5	6	7	8	9	10	11	12

Here, we have checked the first, fifth, and tenth boxes. Each such diagram corresponds to a way of investing the remaining money as follows. We order the mutual funds. Count the number of unchecked boxes to the left of the first checkmark. Call this number k_1. In the illustration above, $k_1 = 0$. Next, count the number of unchecked boxes between the first two checkmarks. Call this number k_2. In the illustration, $k_2 = 3$. Next, call k_3 the number of unchecked boxes between the second and third checkmarks, and call k_4 the number of unchecked boxes after the third checkmark. Thus, $k_3 = 4$ and $k_4 = 2$. Observe that $k_1 + k_2 + k_3 = 9$, the total number of unchecked boxes. Make additional investments (beyond the required minimal investments) of k_1 thousand dollars in the first fund, k_2 thousand dollars in the second fund, k_3 thousand dollars in the third fund, and k_4 thousand dollars in the fourth fund. Thus, the total number of different investments is the same as the number of ways of choosing three blocks (to check) from among 12 blocks. This number is

$$\binom{12}{3} = \frac{12!}{3! \times 9!} = \frac{12 \times 11 \times 10}{3 \times 2 \times 1} = 220.$$

(24) (c) There are $n(n-1)/2$ different ways of choosing 2 objects from a given collection of n objects. Thus, there are $10 \times 9/2 = 45$ different ways of choosing 2 of the 10 coins. For each selection of two coins, one can form 4 tangent lines as shown at the right. Therefore, the answer is $4 \times 45 = 180$.



(25) (e) Let f_n denote the number of ways one can get to the nth step. Given the example in the problem, we see then that $f_3 = 3$. It is easy to see $f_1 = 1$ and $f_2 = 2$ (the latter follows since one can take one step twice or two steps at a time to get to the second step). To determine the value of f_n for any $n \geq 3$, think about the previous step a person would have to be on before getting to the nth step. Since only one or two steps are allowed at a time, the person would have to be either on the $(n-1)$st step or the $(n-2)$nd step. Since

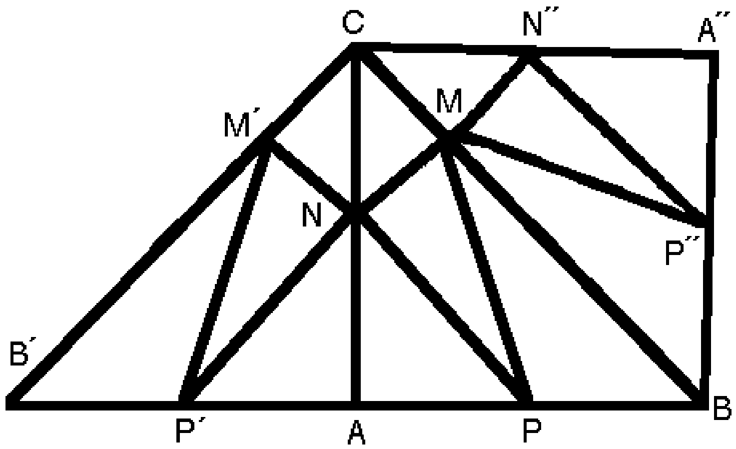
there are by definition f_{n-1} ways someone can get to the $(n-1)$st step and f_{n-2} ways a person can get to the $(n-2)$nd step, there must be $f_{n-1}+f_{n-2}$ ways of getting to the nth step (by first going to the $(n-1)$st step in one of f_{n-1} ways and then taking a single step or by first going to the $(n-2)$nd step in one of f_{n-2} ways and then taking two steps at once). In other words, $f_n = f_{n-1}+f_{n-2}$. This allows us to easily compute f_n for $n \geq 3$ successively. Specifically,

$$f_3 = f_2 + f_1 = 2+1 = 3 \text{ (as we knew)}, \quad f_4 = f_3 + f_2 = 3+2 = 5,$$

$$f_5 = f_4 + f_3 = 5+3 = 8, \quad \text{and} \quad f_6 = f_5 + f_4 = 8+5 = 13.$$

Therefore, the answer is 13. (Comment: The numbers f_n are called Fibonacci numbers.)

(26) (d) Rotate the given figure about the line segment $\overline{AC}$ to form $\Delta AB'C \cong \Delta ABC$ and $\Delta P'M'N \cong \Delta PMN$. Now, rotate the given figure about $\overline{BC}$ to obtain $\Delta A''BC \cong \Delta ABC$ and $\Delta P''MN'' \cong \Delta PMN$ so that we get the figure to the right. Observe that the perimeter of ΔPMN is the same as the sum of the lengths of the segments $\overline{P'N}$, $\overline{NM}$, and $\overline{MP''}$. Also, observe that the locations of the points P' and P'' are independent of the selection of the points N on side $\overline{CA}$ and M on side $\overline{BC}$. Thus, the perimeter of ΔPMN or, in other words, the sum of the lengths of the segments $\overline{P'N}$, $\overline{NM}$, and $\overline{MP''}$, is minimized when we force the polygonal path from P' to N to M and then to P'' to be a straight line (the shortest distance between two points on a plane is a straight line). The conditions in the problem imply that $\angle ABC$ is a $45°$ angle so that $\angle PBP''$ is a right angle. It follows that $\Delta P'BP''$ is a right triangle. Its legs have lengths 6 and 2 (by the conditions in the problem). Therefore, the distance from P' and P'' is $\sqrt{6^2+2^2} = \sqrt{40}$. This then is the answer.



(27) (b) Observe that

$$(x+1)^k - x^k = kx^{k-1} + \text{(terms of smaller degree)}.$$

It follows that if $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with $n \geq 1$ and $a_n \neq 0$, then

$$\begin{aligned} f(x+1) - f(x) &= a_n\big((x+1)^n - x^n\big) + a_{n-1}\big((x+1)^{n-1} - x^{n-1}\big) + \cdots + a_1\big((x+1) - x\big) \\ &= a_n\big(nx^{n-1} + \text{(smaller degree terms)}\big) \\ &\qquad + a_{n-1}\big((n-1)x^{n-2} + \text{(smaller degree terms)}\big) + \cdots + a_1. \end{aligned}$$

Hence, $f(x+1)-f(x)$ is a polynomial of degree $n-1$ with leading coefficient $na_n \neq 0$. Since $f_1(x)$ is given as a polynomial of degree $2+6+10=18$, it follows that $f_2(x) = f_1(x+1)-f_1(x)$ is of degree 17, $f_3(x) = f_2(x+1)-f_2(x)$ is of degree 16, $\ldots$, $f_{18}(x)$ is of degree 1, and $f_{19}(x)$ is of degree 0 (i.e., $f_{19}(x)$ is a constant). The condition $f_n(1) = f_n(2) = \cdots = f_n(25)$ implies that $f_n(x)-f_n(1)$ has roots $1, 2, \ldots, 25$. Since a non-zero polynomial cannot have more roots than its degree and each $f_j(x)$ has degree ≤ 18, we must have that $f_n(x)-f_n(1)=0$. In other words, $f_n(x) = f_n(1)$ for all x. This means that $f_n(x)$ is a constant. It is also clear that if $f_n(x)$ is a constant, then $f_n(1) = f_n(2) = \cdots = f_n(25)$. Therefore, we are left with determining the first n for which $f_n(x)$ is constant. As indicated above, this n is 19.

(28) (c) Let d_j denote the jth digit of $1/1996$ after the decimal. In other words, we write

$$\frac{1}{1996} = 0.d_1d_2d_3\ldots.$$

The problem is to determine d_{46}. Observe that

$$(*) \qquad 10^{45} \times \frac{1}{1996} = d_1d_2d_3\ldots d_{45}.d_{46}d_{47}\ldots.$$

We compute 10^{45} modulo 1996. Since $1996 \times 5 = (2000-4) \times 5 = 10^4 - 20$, we get that $10^4 \equiv 20 \pmod{1996}$. Since $1996 \times 4 = (2000-4) \times 4 = 8000 - 16$, we obtain $10^{12} \equiv (10^4)^3 \equiv 20^3 \equiv 8000 \equiv 16 \pmod{1996}$. Continuing in this manner, we get $10^{36} \equiv (10^{12})^3 \equiv 16^3 \equiv 2^{12} \equiv 4096 \equiv 104 \pmod{1996}$, $10^{44} \equiv 10^{36} \times 10^4 \times 10^4 \equiv 104 \times 20 \times 20 \equiv 2080 \times 20 \equiv 84 \times 20 \equiv 1680 \pmod{1996}$, and $10^{45} \equiv 10^{44} \times 10 \equiv 16800 \equiv 8 \times (1996 + 4) + 800 \equiv 832 \pmod{1996}$. This means that the remainder when we divide 10^{45} by 1996 is 832. If q denotes the quotient, we deduce that

$$\frac{10^{45}}{1996} = q + \frac{832}{1996} = q.4168\ldots,$$

where the digits after the decimal are computed by long division. We are only interested in the first of these, the digit 4 since it follows from $(*)$ that this digit must be d_{46}.

(29) (c) The answer is 3^3. Set

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{581}.$$

We will show that there are integers N and M that are not divisible by 3 for which $3^3(2N)S = M$. Thus, $S = M/(3^3(2N))$ and since N and M are not divisible by 3, this fraction when reduced will be such that 3^3 is the largest power of 3 dividing the denominator.

We divide the sum defining S into three subsums S_1, S_2, and S_3 as follows. The sum S_1 is taken over the numbers $1/m$ where 81 divides m (the largest being $567 = 3^4 \times 7$), the sum S_2 is taken over the remaining numbers $1/m$ with $m \leq 566$, and the sum S_3 is taken over the remaining numbers $1/m$ with $m > 567$. Observe that

$$\begin{aligned} S_1 &= \frac{1}{81} + \frac{1}{2\times 81} + \cdots + \frac{1}{7\times 81} \\ &= \frac{1}{81} + \frac{1}{3}\left(\frac{1}{81} + \frac{1}{2\times 81}\right) + \left(\frac{1}{2\times 81} + \frac{1}{7\times 81}\right) + \left(\frac{1}{4\times 81} + \frac{1}{5\times 81}\right) \\ &= \frac{1}{81} + \frac{1}{2\times 81} + \frac{u_1}{3^2\times v_1} = \frac{1}{2\times 3^3} + \frac{u_1}{3^2\times v_1} \end{aligned}$$

where u_1 and v_1 denote integers with v_1 not divisible by 3. For S_2, we have the following:

$$\begin{array}{rccccccccc} 2S_2 = & 1 & + & \frac{1}{2} & + & \frac{1}{3} & + \cdots + & \frac{1}{566} \\ + & \frac{1}{566} & + & \frac{1}{565} & + & \frac{1}{564} & + \cdots + & 1 \\ \hline \end{array}$$

a sum of numbers of the form $567/m$
where 3^7 does not divide m

Note: The display to the left is a bit misleading. Only the terms $1/m$ with m not divisible by 81 appear.

Since $567 = 3^4 \times 7$, it follows that $S_2 = \dfrac{u_2}{3^2\times v_2}$ for some integers u_2 and v_2 with v_2 not divisible by 3. Finally, S_3 is a sum of numbers $1/m$ where each m in the sum is not divisible by 3^3, so we have $S_3 = \dfrac{u_3}{3^2\times v_3}$ for some integers u_3 and v_3 with v_3 not divisible by 3. Let $N = v_1v_2v_3$. We deduce that there are integers k_1, k_2, and k_3 such that

$$3^3(2N)S_1 = N + 3k_1, \quad 3^3(2N)S_2 = 3k_2, \quad \text{and} \quad 3^3(2N)S_3 = 3k_3.$$

Hence, $3^3(2N)S = N + 3(k_1+k_2+k_3)$. Since N is not divisible by 3, this gives us what we set out to show (with $M = N + 3(k_1+k_2+k_3)$).

(30) (b) Every positive integer has a unique representation in base 2. This is the same as saying that each positive integer can be written uniquely as a sum of different powers of 2. For example, $1 = 2^0$, $2 = 2^1$, $3 = 2^0 + 2^1$, and so on. Recall that we are interested in the numbers beginning $x_1 = 3^0$, $x_2 = 3^1$, and $x_3 = 3^0 + 3^1$. We can find the nth term in this sequence by writing n as a sum of different powers of 2 and then replacing each 2^j in the sum by 3^j. Since $n = 100$ can be written in base 2 as $n = (1100100)_2$, we get

$$100 = 2^2 + 2^5 + 2^6 \quad \text{so that} \quad x_{100} = 3^2 + 3^5 + 3^6 = 9 + 243 + 729 = 981.$$