SOLUTIONS TO THE USC MATHEMATICS CONTEST 1998

(1) (a) The numbers between 100 and 1000 that are divisible by 7 are of the form $98+7k$ where k is a positive integer satisfying $98 + 7k \leq 1000$. In other words, k is a positive integer $\leq (1000-98)/7 = 902/7$ (equivalently, $1 \leq k \leq 128$). Hence, the answer is 128.

(2) (c) If each of the four friends caught ≥ 3 fish, then there would have to have been a total of at least $4 \times 3 = 12$ fish caught. So (c) must be true. To see that the others are not necessarily true, note that (a), (b), and (e) would not hold if one person caught 8 fish and the other three caught one fish each; furthermore, (d) would not hold if three of the friends caught three fish each and the fourth friend caught two fish.

(3) (e) If $T = (-2, 3)$, then the slopes of $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RT}$ are the same (namely, -2) and the slopes of $\overleftrightarrow{PR}$ and $\overleftrightarrow{QT}$ are the same (namely, $-1/2$). So (e) is a correct answer. The other choices for T can be seen not to produce parallelograms in a similar manner by considering slopes of the edges they would produce.

(4) (c) If the equations hold for (x, y), then

$$y^2 - y = (x + y^2) - (x + y) = 4 - 2 = 2.$$

Thus, $y^2 - y - 2 = 0$ so that $(y-2)(y+1) = 0$. We deduce that $y = 2$ (and so $x + y = 2$ implies $x = 0$) or $y = -1$ (and so $x+y = 2$ implies $x = 3$). Thus, $(0, 2)$ and $(3, -1)$ are the only possible pairs (x, y) satisfying the given system of equations. One checks that these two possibilities are solutions to the equations, so there are exactly 2 pairs (x, y) as in the problem.

(5) (d) The number $x^2 + x^3 = x^2(x + 1)$ is the square of an integer if and only if $x + 1$ is a square (as can be seen by considering the prime factorization of $x^2(x + 1)$). In other words, we want to determine the number of $x \in \{1, 2, \ldots, 100\}$ such that $x + 1 = n^2$ for some positive integer n. This is equivalent to finding the number of positive integers n for which $2 \leq n^2 \leq 101$ which is easily seen to be 9.

(6) (c) If we expand $(x^2 + ax + 1)(x^2 + bx + 2)$, we see that the coefficient of x^3 is $a + b$ and the coefficient of x is $2a + b$. The equation in the problem implies $a + b = 2$ and $2a+b = -1$. Solving for a and b, we get $a = -3$ and $b = 5$. The value of k is the coefficient of x^2 in $(x^2 - 3x + 1)(x^2 + 5x + 2)$, which is -12.

(7) (a) Suppose the price of the car which made a 40% profit is x dollars and the price of the car which made a 20% loss is y dollars. Since each sold for \$560, we deduce that $(1 + 0.40)x = 560$ and $(1 - 0.20)y = 560$. The dealer's net profit in dollars is

$$560 \times 2 - (x + y) = 1120 - \frac{560}{1.4} - \frac{560}{0.8} = 1120 - 400 - 700 = 20.$$

(8) (a) The side opposite the angle α has length $\sqrt{3}$, and the hypotenuse of the triangle is $\sqrt{(\sqrt{3})^2 + 2^2} = \sqrt{7}$. Hence, $\sin\alpha = \sqrt{3}/\sqrt{7} = \sqrt{3/7}$.

(9) (c) Each throw of the two coins, say A and B, can produce one of four equally likely results: A and B can both show heads, A can show heads and B can show tails, A can show tails and B can show heads, or A and B can both show tails. The player loses when one of these last three outcomes occurs on each of the four throws. We deduce that the player loses with probability

$$\frac{3\times 3\times 3\times 3}{4\times 4\times 4\times 4}=\frac{81}{256},$$

this fraction denoting the total number of possible outcomes in which a loss can occur divided by the total number of possible outcomes for the four throws of the coins. The player wins then with probability $1-(81/256)=175/256\approx 68\%$.

(10) (c) The distance from the center of the equilateral triangle to each vertex is 10. A line from the center of the equilateral triangle to a vertex bisects the angle at that vertex forming two angles with the sides of the triangle, each measuring $30°$. Using that a triangle with hypotenuse 10 and an angle measuring $30°$ has its legs of length 5 and $5\sqrt{3}$, we obtain that the height of the triangle is 15 and one-half the length of a side is $5\sqrt{3}$. The area is therefore $15\times 5\sqrt{3}=75\sqrt{3}$.

(11) (d) The inequality $0<\sin 12°<\cos 12°<1$ implies that the choices (a), (b), and (c) are all <1, that choices (d) and (e) are >1, and that choice (d) is larger than choice (e). (The same reasoning would apply with $12°$ replaced by any angle θ measuring strictly between $0°$ and $45°$.)

(12) (c) Using properties of exponents, observe that $y=2^{2^x}$ implies $y^2=4^{2^x}$. It follows then that $y^2+y-42=0$. On the other hand, $y^2+y-42=(y+7)(y-6)$. Since $y=2^{2^x}$ is clearly positive, we deduce that $y=6$. The problem is equivalent to determining the value of $\sqrt{2^y}=\sqrt{2^6}$, which is 8.

(13) (b) Let $\alpha=\sin(\pi/7)$ and $\beta=\cos(\pi/7)$, and note that $\alpha^2+\beta^2=1$. The conditions in the problem imply

$$x^2-bx+c=(x-\alpha)(x-\beta)=x^2-(\alpha+\beta)x+\alpha\beta.$$

We deduce that

$$b^2=(\alpha+\beta)^2=\alpha^2+\beta^2+2\alpha\beta=1+2c.$$

(14) (c) The distance from B to C is 2. We translate the point $D=(-5,-3)$ up two units to $D'=(-5,-1)$. The shortest path as in the problem is equal in length to 2 (the distance from B to C) plus the length of the shortest path from D' to A. The shortest path from D' to A is along a straight line, so its length is the distance from $D'=(-5,-1)$ to $A=(7,4)$. This distance is $\sqrt{(7+5)^2+(4+1)^2}=13$. Therefore, the answer is $2+13=15$.

(15) (b) Let x represent the size of the bottle in gallons. Initially, x gallons of pure alcohol are emptied into B and stirred. At this point, B contains a mixture that has a ratio of alcohol to total gallons of mixture of $x/(6+x)$. Then x gallons of this mixture are emptied into A so that A once again has 6 gallons of liquid. The amount of alcohol in the mixture of

alcohol and water in the bottle that was poured into A is $x \times x/(6+x) = x^2/(6+x)$ gallons. Therefore, A ends up with $6 - x + (x^2/(6+x))$ gallons of alcohol and $x - (x^2/(6+x))$ gallons of water. The conditions in the problem imply

$$\frac{6 - x + (x^2/(6+x))}{x - (x^2/(6+x))} = \frac{4}{1}.$$

Rewriting the above and solving for x gives $x = 3/2 = 1.5$.

(16) (a) Since $P(2) = P(-2) = P(-3) = -1$, each of the numbers 2, -2, and -3 is a root of $P(x) + 1$. Given that $P(x)$ is a polynomial of degree 4, we deduce that there are numbers a and b such that

$$P(x) = (x-2)(x+2)(x+3)(ax+b) - 1.$$

Using that $P(1) = P(-1) = 1$, we obtain $-12a - 12b - 1 = 1$ and $6a - 6b - 1 = 1$. Solving for a and b, we get that $a = 1/12$ and $b = -1/4$ (and $P(x) = ((x-2)(x+2)(x+3)(x-3)/12) - 1)$. We deduce that $P(0) = -12b - 1 = 2$.

(17) (e) Let t denote the number of juniors (so t also is the number of seniors). Let y be the number of respondents that answered "Yes", and let n be the number of respondents that answered "No". Then $0.60y = 3y/5$ is the number of seniors that said "Yes" and $0.80n = 4n/5$ is the number of juniors that said "No". The number of juniors that said "Yes" is $2y/5$ so that $(2y/5) + (4n/5) = t$. The number of seniors that said "No" is $n/5$ so that $(3y/5) + (n/5) = t$. Thus,

$$2y + 4n = 5t \quad \text{and} \quad 3y + n = 5t.$$

It follows that $y = 3t/2$ (and $n = t/2$). The proportion of juniors that said "Yes" is $(2y/5)/t = 3/5 = 60\%$.

(18) (b) Let $N = 99999899999$. Then $N = 10^{11} - 10^5 - 1$. Thus,

$$\begin{aligned} N^2 &= (10^{11} - 10^5 - 1)^2 \\ &= 10^{22} - 2 \times 10^{16} - 2 \times 10^{11} + 10^{10} + 2 \times 10^5 + 1 \\ &= 9999979999810000200001. \end{aligned}$$

The digit 9 appears nine times in the decimal expansion of N^2.

(19) (c) From the given information, $\alpha = \tan^{-1}(1/5)$ and $\beta = \tan^{-1}(3/2)$. Observe also that $0 < \beta - \alpha < 90°$. We can obtain $\beta - \alpha$ by computing

$$\tan(\beta - \alpha) = \frac{\tan\beta - \tan\alpha}{1 + (\tan\alpha)(\tan\beta)} = \frac{(3/2) - (1/5)}{1 + (3/2)(1/5)} = 1.$$

We deduce that $\beta - \alpha = 45°$.

(20) (b) Let n denote the unique positive integer in the problem. We give two approaches to obtaining n. First, note that since n ends in a 6, $4n$ ends in a 4. This means the last two digits of n are 46, so the last two digits of $4n$ are 84. Hence, the last three digits of n are 846, and the last three digits of $4n$ are 384. Continuing this way, we obtain quickly that the last six digits of n are 153846 and that the last six digits of $4n$ are 615384. At this point, we realize we have the solution to the problem, namely $n = 153846$ (since then $4n$ is of the form indicated in the statement of the problem).

For the second approach, we write n in the form $n = 10m + 6$ where m is some positive integer. If r is the number of digits in m, then the conditions in the problem give that $6 \times 10^r + m = 4n = 40m + 24$. Thus, $6 \times 10^r = 39m + 24$. Dividing by 3 and then working modulo 13, we deduce that $2 \times 10^r \equiv 8 \pmod{13}$ so that $10^r \equiv 4 \pmod{13}$. One computes the congruences

$$\begin{aligned}
10^1 &\equiv 10 \pmod{13} \\
10^2 &\equiv 100 \equiv 9 \pmod{13} \\
10^3 &\equiv 90 \equiv 12 \pmod{13} \\
10^4 &\equiv 120 \equiv 3 \pmod{13} \\
10^5 &\equiv 30 \equiv 4 \pmod{13}.
\end{aligned}$$

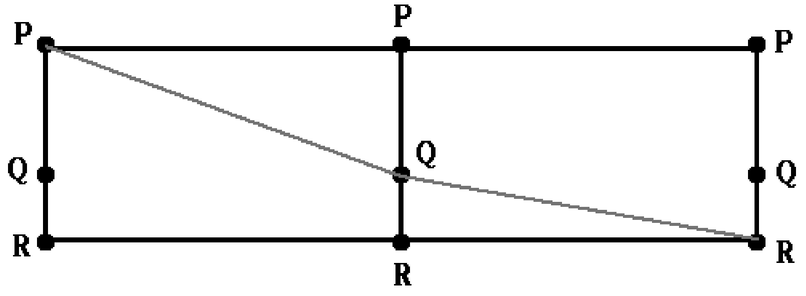
So $r = 5$ is a reasonable guess (and this would give the smallest possible n). Taking $r = 5$ in $6 \times 10^r = 39m + 24$ and solving for m, we obtain $m = (600000 - 24)/39 = 15384$. Corresponding to this value of m, one checks that $n = 153846$ is in fact a number as in the problem.

(21) (e) Let $f(x) = x^{200} - 2x^{199} + x^{50} - 2x^{49} + x^2 + x + 1$. Observe that $f(1) = 1$ and $f(2) = 7$. Let $q(x)$ and $r(x)$ denote the quotient and remainder, respectively, when $f(x)$ is divided by $(x-1)(x-2)$. Since $(x-1)(x-2)$ is a quadratic, we deduce that $r(x) = Ax+B$ for some numbers A and B. Thus,

$$f(x) = (x-1)(x-2)q(x) + Ax + B.$$

We deduce that $f(1) = A+B$ and $f(2) = 2A+B$. Since $f(1) = 1$ and $f(2) = 7$, we obtain $A + B = 1$ and $2A + B = 7$. We deduce that $A = 6$ and $B = -5$. Hence, $r(x) = 6x - 5$.

(22) (a) We cut the can along a line passing through P, Q, and R, and we stretch out the can so that its side is in the shape of a rectangle. Then P, Q, and R appear as points on two opposite edges of the rectangle. The length of the rectangle is 4π, and its height is 6. We draw two such rectangles beside each other and indicate where P, Q, and R are along



the edges as shown. The shortest path from P to Q to R is indicated above (the first time around the can is indicated by the line segment from P to Q the second time around the can is indicated by the line segment from Q to R). Given that the vertical distance from P to Q is 4 and the vertical distance from Q to R is 2, we get from two applications of the Pythagorean theorem that the length of the shortest path is

$$\sqrt{(4\pi)^2+4^2}+\sqrt{(4\pi)^2+2^2}=4\sqrt{\pi^2+1}+2\sqrt{4\pi^2+1}.$$

(23) (b) If $p=3$, then $p^2+21p-1=71$ which is prime. Otherwise, p is not divisible by 3 (since it is prime) and the remainder when p is divided by 3 is either 1 or 2. In other words, either $p=3k+1$ for some integer k or $p=3k+2$ for some integer k. In either case, $k\ge 0$. Also, in the first case, $p^2+21p-1=9k^2+6k+21p=3(3k^2+2k+7p)$; and, in the second case, $p^2+21p-1=9k^2+12k+3+21p=3(3k^2+4k+1+7p)$. Thus, if p is a prime other than 3, then $p^2+21p-1$ is an integer >3 which is divisible by 3 (and, therefore, $p^2+21p-1$ is not prime).

(24) (e) There are 201 vertical lines and 201 horizontal lines. Each vertical line and each horizontal line together intersect at exactly one point, so there are 201^2 intersection points arising from these lines. For every integer $k\in[0,100]$, each of the lines $x=\pm k$ and $y=\pm k$ intersects the circles centered at the origin with radii $k+(1/\pi), k+1+(1/\pi),\dots,99+(1/\pi)$ and no others. For each such line and each such circle, there are precisely two points of intersection. Therefore, for each integer $k\in[1,100]$, there are precisely $8(100-k)$ points of intersection among the lines $x=\pm k$ and $y=\pm k$ and the given circles. For $k=0$, there are precisely $4(100-k)=400$ points of intersection among the lines $x=\pm k=0$ and $y=\pm k=0$ and the given circles. The parallel lines do not intersect among themselves and the same is true of the concentric circles. Hence, the total number of intersection points is

$$\begin{aligned}201^2+400+\sum_{k=1}^{100}8(100-k)&=40401+400+\sum_{k=1}^{100}800-8\sum_{k=1}^{100}k\\&=40801+800\times 100-8\times\frac{100\times 101}{2}=80401.\end{aligned}$$

(25) (a) Let c denote the third root of $P(x)=x^3+3x^2-1$. Clearly, $P(0)=-1$. Since $P(x)=(x-a)(x-b)(x-c)$, it also follows that $P(0)=-abc$. Therefore, $c=1/(ab)$. Since c is a root of $P(x)$, we obtain $1/(ab)^3+3/(ab)^2-1=0$. Multiplying by $-(ab)^3$ gives $-1-3(ab)+(ab)^3=0$, so ab is a root of x^3-3x-1.

An alternative approach to obtaining this cubic is to consider $(x-ab)(x-ac)(x-bc)$, which clearly has ab as a root. Observe that

$$(x-ab)(x-ac)(x-bc)=x^3-(ab+ac+bc)x^2+(a^2bc+ab^2c+abc^2)x-a^2b^2c^2.$$

On the other hand, since a, b, and c are roots of x^3+3x^2-1, we have

$$x^3+3x^2-1=(x-a)(x-b)(x-c)=x^3-(a+b+c)x^2+(ab+ac+bc)x-abc.$$

This implies $a+b+c=-3$, $ab+ac+bc=0$, and $abc=1$. Thus, $a^2bc+ab^2c+abc^2=abc(a+b+c)=-3$ and $a^2b^2c^2=(abc)^2=1$. It follows that $(x-ab)(x-ac)(x-bc)=x^3-3x-1$.

With either of the above approaches, we see that (a) is a correct answer. The possibility that one of the other answers is also correct should be eliminated. We illustrate how this can be done. Let $f(x)=x^3-3x-1$. Then ab is a root of $f(x)$. Assume ab is also a root of x^3+x^2-3x+1 (choice (b)). Then ab is a root of $x^3+x^2-3x+1-f(x)=x^2+2$ and, therefore, of $x(x^2+2)-f(x)=5x+1$. But then ab is a root of $25(x^2+2)-(5x-1)(5x+1)=51$. But clearly nothing can be a root of the polynomial 51, so our assumption was wrong and ab is not a root of x^3+x^2-3x+1. A similar argument can be given to show that each of the remaining choices is also not a correct answer. (The approach used here is an application of a variation on the Euclidean algorithm.)

(26) (d) Squaring both sides of the equation $\sqrt{x}=\sqrt{1998}-\sqrt{y}$ gives $x=1998-2\sqrt{1998y}+y$. Using that $1998=2\times 3^3\times 37$, we deduce that $\sqrt{2\times 3\times 37\times y}=\sqrt{222y}$ is a rational number. It follows that $y=222m^2$ for some nonnegative integer m (see the comment below). A similar argument gives that $x=222n^2$ for some nonnegative integer n. Since $\sqrt{1998}=3\sqrt{222}$, the equation $\sqrt{x}+\sqrt{y}=\sqrt{1998}$ implies that $m+n=3$. One obtains 4 solutions for (x,y) corresponding to (m,n) belonging to $\{(0,3),(1,2),(2,1),(3,0)\}$.

We have used that if k is an integer for which $\sqrt{k}$ is rational, then k is the square of an integer. This can be seen by noting that if $\sqrt{k}=a/b$ for some integers a and b, then $b^2k=a^2$ so that the prime factorization of b^2k and a^2 must be the same. The latter implies that the highest power of any prime p dividing k must be of the form p^r where r is an even integer. Thus, k must be a square.

(27) (e) Since $xyz=4000$, the only prime divisors of x, y, and z are 2 and 5. Writing $x=2^a5^d$, $y=2^b5^e$, and $z=2^c5^f$ (where each exponent is a nonnegative integer), we have

$$2^{a+b+c}5^{d+e+f}=xyz=4000=2^55^3.$$

Thus, we want to know the number of solutions in nonnegative integers to $a+b+c=5$ and $d+e+f=3$. For each $a\in\{0,1,2,3,4,5\}$, the number of b and c such that $b+c=5-a$ is $5-a+1$ (since b can be any one of $0,1,\dots,5-a$ and then c is the unique nonnegative integer given by $c=5-a-b$). Thus, the number of triples (a,b,c) for which $a+b+c=5$ is

$$\sum_{a=0}^{5}(5-a+1)=6+5+4+3+2+1=21.$$

Similarly, the number of triples (d,e,f) such that $d+e+f=3$ is

$$\sum_{d=0}^{3}(3-d+1)=4+3+2+1=10.$$

Therefore, the total number of possibilities for all 6 numbers a, b, c, d, e, and f (and, hence, the number of possibilities for x, y, and z) is $21\times 10=210$.

We note that there is another approach to counting the number of nonnegative integer solutions of $a+b+c=5$. Consider seven successive blanks numbered as follows:

$$\underset{1}{\_}\ \underset{2}{\_}\ \underset{3}{\_}\ \underset{4}{\_}\ \underset{5}{\_}\ \underset{6}{\_}\ \underset{7}{\_}$$

If we put a check mark on any two of them, we obtain a choice for a, b, and c corresponding to the number of blanks before the first check mark (for a), the number of blanks between the two check marks (for b), and the number of blanks after the second check mark (for c). For example, if the blanks numbered 3 and 4 are checked, then we would have $a=2$, $b=0$, and $c=3$. This gives a total of $\binom{7}{2}=\frac{7!}{2!\,5!}=21$ different possibilities for a, b, and c, as before. A similar argument shows that, more generally, if k and n are positive integers, then the number of solutions in nonnegative integers $a_1, a_2, \dots, a_k$ to $a_1+a_2+\cdots+a_k=n$ is $\binom{n+k-1}{k-1}$.

(28) (d) In the drawing, put the y-axis along the line $\overleftrightarrow{AB}$ and put the x-axis along the line $\overleftrightarrow{BC}$. Then B is at the origin $(0,0)$. The shaded region is bounded by four lines, one through each of the points $A=(0,1)$, $B=(0,0)$, $C=(1,0)$, and $D=(1,1)$. The line through A also passes through $(2/3,0)$ so its equation is $y=(-3/2)x+1$. The line through C is parallel and so has the same slope; its equation is $y=(-3/2)x+(3/2)$. The line through B passes through $(1,2/3)$ and has equation $y=(2/3)x$. The line through D is parallel to the line through B. Since the slope of the line through A is minus the reciprocal of the slope of the line through B, these lines are perpendicular. It follows that the shaded region is a rectangle. In fact, because of the symmetry in the problem, each edge of the rectangle will have the same length so that the rectangle is a square. To find its area, we determine the length of a side of the square. The lines through A and B (along two sides of the inner square) intersect at the point $(6/13,4/13)$ (which is obtained from the equations of these lines above). Also, the lines through B and C (along two sides of the inner square) intersect at the point $(9/13,6/13)$. The distance from $(6/13,4/13)$ to $(9/13,6/13)$ is $\sqrt{(3/13)^2+(2/13)^2}=1/\sqrt{13}$. Thus, the area of the shaded region is $(1/\sqrt{13})^2=1/13$.

(29) (d) Observe that

$$f(3)=\log_2(\log_3 3)-\log_3(\log_2 3)=\log_2 1-\log_3(\log_2 3)=-\log_3(\log_2 3)<0.$$

We make use of the change of base formula $\log_b a=\dfrac{\log_c a}{\log_c b}$. Using this change of base formula and other basic properties of logarithms, we deduce

$$\begin{aligned} f(x)=\log_2(\log_3 x)-\log_3(\log_2 x)&=\log_2\left(\frac{\log_2 x}{\log_2 3}\right)-\frac{\log_2\log_2 x}{\log_2 3}\\ &=\log_2\log_2 x-\log_2\log_2 3-\frac{\log_2\log_2 x}{\log_2 3}=\left(1-\frac{1}{\log_2 3}\right)\log_2\log_2 x-\log_2\log_2 3. \end{aligned}$$

Since the definition of the logarithm implies $\log_2 x$ is an increasing function for $x>0$ which tends to infinity with x, we obtain that $\log_2\log_2 x$ is increasing for $x>1$ and tends to infinity with x. Thus, $f(3) < 0$, $f(x)$ is increasing for $x \geq 3$, and $f(x)$ tends to infinity with x. These imply (d) is true and each of (a), (b), (c), and (e) is not true.

(30) (d) Observe that

$$\frac{a}{b} = \frac{1997}{1998} + \frac{1999}{n} = \frac{1997n + 1998 \times 1999}{1998n}.$$

Since a is divisible by 1000, a is even. It follows that n must be even. We write $n = 2m$ and simplify the above expression for a/b to obtain

$$\frac{a}{b} = \frac{1997m + 999 \times 1999}{1998m}.$$

Since a is even, we see that m must be odd. Since 5 divides a, we also see that m cannot be divisible by 5. So we consider m now having no prime divisors in common with 10. Observe that the denominator on the right-hand side above is divisible by 2 and not 4. Also, this denominator is not divisible by 5. Thus, in order for a to be divisible by 1000, it is necessary and sufficient for $1997m + 999 \times 1999$ to be divisible by 2000. We solve for m by working modulo 2000. We seek m for which

$$0 \equiv 1997m + 999 \times 1999 \equiv -3m + 999(-1) \pmod{2000}$$

which is equivalent to

$$m \equiv -333 \equiv 1667 \pmod{2000}.$$

(Here, we have used that 3 and 2000 have no common prime divisors so that division by 3 in a congruence modulo 2000 is permissible.) Note that $m \equiv 1667 \pmod{2000}$ implies that m and 10 have no common prime divisors. Hence, the condition $m \equiv 1667 \pmod{2000}$ is a necessary and sufficient condition for $n = 2m$ to result in a fraction a/b as in the problem with a divisible by 1000. Therefore, the smallest such positive integer is $2 \times 1667 = 3334$. The sum of its digits is 13.