SOLUTIONS TO THE USC MATHEMATICS CONTEST 1997

(1) (a) One groups the first two terms and each successive two terms to obtain a sum of 500 expressions each of which is -1. The answer is -500.

(2) (a) If x and y denotes the two numbers, then $(x+y)^2 = (x-y)^2+4xy = 2^2+4\times 17 = 72$. Since x and y are positive, the answer is $\sqrt{72} = 6\sqrt{2}$.

(3) (b) Since 1 is a root of x^3+5x^2-2x-4, we know that $x-1$ is a factor. Factoring, we obtain $x^3+5x^2-2x-4 = (x-1)(x^2+6x+4)$. The other solutions of the given equation are the roots of x^2+6x+4 which, by the quadratic formula, are $-3+\sqrt{5}$ and $-3-\sqrt{5}$. The answer is (b).

(4) (d) The graph intersects the x-axis at the real roots of $x-2$, $2x^2-5x+4$, and $2x^2-7x+4$. Using the quadratic formula, one can determine the roots of the quadratics. The first quadratic has imaginary roots. The second has roots $(7\pm\sqrt{17})/4$. Therefore, the graph intersects the x-axis at 2, $(7+\sqrt{17})/4$, and $(7-\sqrt{17})/4$.

(5) (d) You can eat 14 green, 14 red, 12 yellow, 10 blue, 14 brown, and 10 orange candies without eating 15 pieces of one color. One more will force 15 to be of some color. That means you will have eaten 75 candies (assuming you don't get sick first).

(6) (b) The game is a tie unless both Michael and Dave hit the target or both Michael and Dave miss the target. One of these will occur with probability $(0.6)(0.3)+(1-0.6)(1-0.3) = 0.18+0.28 = 0.46$.

(7) (b) Converting all logarithms to the same base, say 10, we obtain that the sum in the problem is equal to

$$\frac{\log_{10}(2)}{\log_{10}(100!)} + \frac{\log_{10}(3)}{\log_{10}(100!)} + \frac{\log_{10}(4)}{\log_{10}(100!)} + \cdots + \frac{\log_{10}(100)}{\log_{10}(100!)}.$$

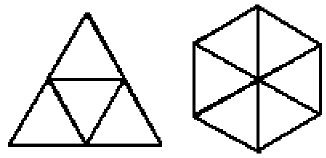
The denominators are all the same and the numerators sum to $\log_{10}(2\times 3\times 4\times\cdots\times 100) = \log_{10}(100!)$. It follows that the above sum is equal to 1.

(8) (c) Let x denote the number of miles per gallon Tim gets during city driving so that $x+3$ is the number of miles per gallon Tim gets during highway driving. Then Tim used $155/x$ gallons in the city and $136/(x+3)$ gallons on the highway. Thus,

$$\frac{155}{x} + \frac{136}{x+3} = 9 \qquad \Longrightarrow \qquad 9x^2 - 264x - 3\times 155 = 0.$$

Factoring a 3 from the last equation gives $3x^2-88x-155 = 0$. Hence, $(3x+5)(x-31) = 0$ so that $x = 31$.

(9) (e) Partitioning the equilateral triangle into 4 equilateral triangles as shown and the regular hexagon into 6 equilateral triangles as shown, we see the answer is $4/6 = 2/3$.



(10) (b) If the legs of the right triangle are a and b, then $294 = a^2 + b^2 + \left(\sqrt{a^2+b^2}\right)^2 = 2\left(a^2+b^2\right)$ so that $a^2 + b^2 = 147$. Hence, $12 + 8\sqrt{3} = a + b + \sqrt{a^2+b^2} = a + b + \sqrt{147} = a + b + 7\sqrt{3}$. We deduce that $a + b = 12 + \sqrt{3}$ and (by squaring) $a^2 + 2ab + b^2 = 147 + 24\sqrt{3}$. Since $a^2 + b^2 = 147$, we obtain that $2ab = 24\sqrt{3}$. The area of the triangle is $ab/2 = 6\sqrt{3}$.

(11) (e) The given information implies that $f(x) = (x-3)(x-13)g(x)$ where $g(x)$ is a polynomial with integer coefficients. Hence, $f(10) = -21g(10)$, so that 21 must be a divisor of $f(10)$. The only choice divisible by 21 is 42, so the correct answer is (e). To see in fact that such an $f(x)$ exists, consider $f(x) = -2(x-3)(x-13)$. (An alternative approach to using the form $(x-3)(x-13)g(x)$ of $f(x)$ is to use that $b - a$ is a factor of $f(b) - f(a)$ for any integers a and b. By considering the pair $(b, a) = (10, 3)$ and using that $f(3) = 0$, one obtains that 7 divides $f(10)$. Considering the pair $(b, a) = (10, 13)$ similarly gives that 3 divides $f(10)$. We deduce that the only possible correct choice is 42.)

(12) (e) Let x be John's walking speed and y be Nancy's walking speed both in feet per minute, and let d denote the distance in feet between the houses. Then in 4 minutes, John travels $4x$ feet and Nancy travels $4y$ feet so that (from the information given) $d = 4x + 4y$. Similarly, one gets $d = 2x + 5y$. It follows that $d = 5d - 4d = 5(4x+4y) - 4(2x+5y) = 12x$. This implies it will take John 12 minutes to travel distance d.

(13) (d) The number 214875 is $3^2 \times 5^3 \times 191$. Since n and m must be 3 digit numbers with product 214875, one can easily check the possible 3 digit factors of 214875 to find one satisfying the conditions in the problem. Alternatively, one could notice that since an integer is divisible by 3 if and only if the sum of its digits is divisible by 3, then n is divisible by 3 if and only if m is. On the other hand, $nm = 214875$ is divisible by 3 so both n and m are. It follows that $3 \times 191 = 573$ is one of n or m and, hence, the middle digit of n must be 7.

(14) (b) That $P(x)$ is a quadratic polynomial satisfying $P(x) \geq 0$ for all x and $P(1) = 0$ implies that $P(x) = c(x-1)^2$ for some constant c (otherwise, the values of $P(1+t)$ and $P(1-t)$ could not both be ≥ 0 if t were sufficiently small). Since $P(2) = 2$, we get $P(x) = 2(x-1)^2$. Therefore, $P(0) + P(4) = 2 + 18 = 20$.

(15) (c) The sum $\sum_{k=0}^{100} a_k$ is simply the number of times one of the numbers $0, 1, 2, \ldots, 100$ occurs in the sequence. On the other hand, each of $a_0, a_1, \ldots, a_{100}$ is one of the numbers $0, 1, 2, \ldots, 100$. Thus, the sum is the number of terms in the sum, namely 101.

(16) (c) The sum of the interior angles of a convex 9-gon is $7 \times 180^\circ$. If d denotes the positive difference (in degrees) between consecutive elements of the arithmetic progression

of interior angles, then $\sum_{k=0}^{8}(112 + kd) = 7 \times 180$. Since $\sum_{k=0}^{8} k = 8 \times 9/2 = 36$, we obtain $112\times 9+36\times d = 7\times 180$. Dividing by 36 and rearranging, we deduce that $d = 35-28 = 7$. Thus, the largest interior angle in degrees is $112 + 8 \times 7 = 168$.

(17) (c) The product modulo 2 is $x^{11}(x+1)^2$. Since this is $x^{13}+x^{11}$ modulo 2, the answer is 2 (the coefficients of x^{13} and x^{11} are odd and all the others are even).

(18) (e) Since $-(a-3)$ is the sum of the roots and a is the product of the roots and since both roots are positive, we get $0 < a < 3$. Since the roots are real and distinct, we obtain that the discriminant of the quadratic is > 0. Hence, $(a-3)^2 - 4a > 0$ so that $a^2 - 10a + 9 > 0$. Since $a^2 - 10a + 9 = (a-1)(a-9)$, we obtain that $a < 1$ or $a > 9$. The condition $0 < a < 3$ now implies $0 < a < 1$ is the only possible correct choice. Observe that $0 < a < 1$ implies that $\sqrt{(a-3)^2 - 4a} < 3 - a$ so that the roots of the quadratic, namely $\big((3-a) \pm \sqrt{(a-3)^2 - 4a}\big)/2$, are positive real numbers.

(19) (e) Observe that d divides $n+d-2$ if and only if d divides $n-2$. Thus, we want the 4^{th} positive integer such that each of $2, 3, \dots, 10$ divides $n-2$. Equivalently, we want the least common multiple of $2, 3, \dots, 10$ to divide $n-2$. Since $\operatorname{lcm}(2, 3, \dots, 10) = 2^3 \times 3^2 \times 5 \times 7 = 2520$, the positive integers n for which $n-2$ is divisible by this least common multiple are those integers of the form $n = 2520k + 2$ where k is a nonnegative integer. Therefore, the 4^{th} such integer is $2520 \times 3 + 2 = 7562$. So $N = 7562$ and the sum of the digits of N is 20.

(20) (d) The given equation $2f(x) + f(1-x) = x^2$ holds for all x. In particular, the equation holds if we replace x with $1-x$. Thus, we deduce that $2f(1-x)+f(x) = (1-x)^2$. Subtracting this equation from twice the given equation results in $3f(x) = 2x^2-(1-x)^2 = x^2 + 2x - 1$. Thus, the answer is (d).

(21) (d) Since $8 \equiv 1 \pmod 7$, we obtain $8^k \equiv 1 \pmod 7$ for every positive integer k. Thus,

$$n \equiv (34112d4357)_8 \equiv 3\times 8^9+4\times 8^8+\cdots+5\times 8+7 \equiv 3+4+\cdots+5+7 \equiv d+30 \equiv d+2 \pmod 7$$

so that, since 7 divides n, we must have $d = 5$.

(22) (d) The binomial theorem implies that the coefficient of x^{39} in $(3x + 5)^{100}$ is $\binom{100}{39}3^{39}5^{61} = (100!/(39!\,61!))3^{39}5^{61}$. Since 2 does not divide $3^{39}5^{61}$, we only need to compute the power of 2 divide each of the factorials. If $[x]$ denotes the greatest integer $\le x$, then the power of a prime p dividing $n!$ is p^r where $r = [n/p]+[n/p^2]+[n/p^3]+\cdots$ (so that r is the number of multiples of p plus the number of multiples of p^2 plus the number of multiples of p^3 and so on). We deduce that if $2^{r(n)}$ divides $n!$, then

$$r(100) = \left[\frac{100}{2}\right] + \left[\frac{100}{4}\right] + \left[\frac{100}{8}\right] + \cdots = 50 + 25 + 12 + 6 + 3 + 1 = 97$$

and, similarly, $r(39) = 19 + 9 + 4 + 2 + 1 = 35$ and $r(61) = 30 + 15 + 7 + 3 + 1 = 56$. Since $r(100) - r(39) - r(61) = 97 - 35 - 56 = 6$, the answer is 2^6. (There is a general result that also could be used that asserts that the power of a prime p which divides $\binom{m}{k}$ is p^r where r denotes the number of borrows needed when subtracting k from m in base p.)

(23) (a) Each of the 4 cards numbered k where $1 \le k \le 10$ can be distributed in a deal to the players in 4! ways. Since k can be any of 10 different numbers, we deduce that there are $4!^{10}$ different deals.

(24) (b) Let θ denote $\angle ABC$, and let r denote the radius of the circle centered at C. We use the Law of Cosines twice. First, since $\cos(\pi/3) = 1/2$, we obtain

$$(r+5)^2 = 8^2 + (r+3)^2 - 16(r+3)\cos(\pi/3) \qquad \Longrightarrow \qquad r = 2.$$

Next, we deduce that

$$5^2 = 8^2 + 7^2 - 112\cos\theta \qquad \Longrightarrow \qquad \cos\theta = 11/14.$$

Hence, the answer is (b).

(25) (b) One can sum the measures of the angles in the non-overlapping triangles by adding the sums of the measures of all the angles at each of the 34 possible vertices for the triangles. For each corner P of the square, the sum of the angles at P is $90°$. For each interior point Q, the sum of the angles at Q is $360°$. Thus, the sum of all the angles of the triangles is $4 \times 90° + 30 \times 360° = 62 \times 180°$. Therefore, there are 62 triangles.

(26) (c) The problem is the same as counting the number of different strings of six 0's and four 1's with no two 1's appearing next to each other. For example, the string 0100100101 corresponds to choosing the numbers 2, 5, 8, and 10 in the problem (the numbers correspond to the positions of the 1's in the string of 0's and 1's). Such strings of 0's and 1's can be obtained by putting 1's in any 4 of the blanks below and bringing the 0's and 1's together.

$$\underset{1}{\_\_}\ 0\ \underset{2}{\_\_}\ 0\ \underset{3}{\_\_}\ 0\ \underset{4}{\_\_}\ 0\ \underset{5}{\_\_}\ 0\ \underset{6}{\_\_}\ 0\ \underset{7}{\_\_}$$

For example, the string 0100100101 above would correspond to putting 1's in the blanks numbered 2, 4, 6, and 7. There are 7 blanks and 4 are to be selected so that the total number of strings as above is $\binom{7}{4} = \dfrac{7!}{3!\,4!} = 35$.

(27) (d) Take $a = x - y$ and $b = 2xy/(x-y)$ in the arithmetic-geometric mean inequality. Since $xy = 2$, we have $b = 4/(x-y)$. Thus,

$$\frac{x^2+y^2}{x-y} = x - y + \frac{2xy}{x-y} = a + b \ge 2\sqrt{ab} = 2\sqrt{4} = 4.$$

Thus, the minimum must be ≥ 4. To see that 4 is possible, we use that equality holds in the arithmetic-geometric mean inequality if and only if $a = b$. Equality occurs then if there are x and y with $x > y > 0$, $xy = 2$, and $x - y = 2xy/(x-y)$. We take $y = 2/x$ (so $xy = 2$) and rewrite $x - y = 2xy/(x-y)$ to deduce that

$$x^2 - 8 + \frac{4}{x^2} = 0 \quad \Longrightarrow \quad (x^2)^2 - 8x^2 + 4 = 0 \quad \Longrightarrow \quad x^2 = \frac{8 \pm \sqrt{64-16}}{2} = 4 \pm 2\sqrt{3}.$$

One takes $x = \sqrt{4+2\sqrt{3}} = \sqrt{3}+1$ and $y = 2/x = \sqrt{3}-1$. Thus, $x > y > 0$, and the value 4 for $(x^2+y^2)/(x-y)$ must occur (with this choice of x and y).

(28) (b) Observe that no indication is given as to where the points P and Q are on sides $\overline{CD}$ and $\overline{BC}$. A (correct) assumption would be that the answer does not depend on where P and Q are placed along these sides (with $\angle PAQ = 45°$). One can guess the answer by considering what happens if P is placed very near to D (or $D = P$) and Q is placed very near to C (or $C = Q$). See Figure 1. A good guess would be that the perimeter of ΔPQC (the sum of the distances PQ, QC, and CP) is twice the distance from C to D or 2.

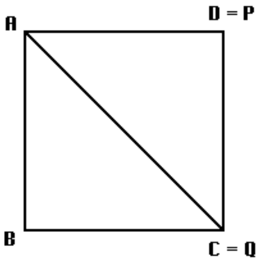


Figure 1

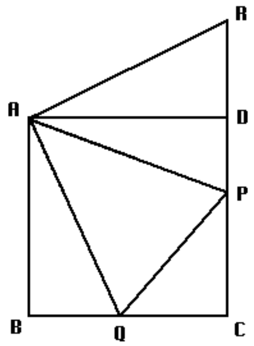


Figure 2

To obtain the answer without any assumptions, consider a point R placed so that ΔADR is congruent to ΔABQ as in Figure 2. Since $\angle PAQ = 45°$, we obtain $\angle RAP = 45°$. Since the distances AR and AQ are the same and the distance AP is equal to itself, we get (by side-angle-side) that ΔAQP is congruent to ΔARP. Hence, $PQ = PR = PD + DR = PD + BQ$. It follows that the perimeter of ΔPQC is $PQ + QC + CP = BQ + QC + CP + PD = 2$.

(29) (e) Call the term in a polynomial with largest degree the "leading term" of the polynomial. We write only the leading terms of the polynomials and use "$\cdots$" for the others. Thus, $f(x) = 7x^3 + \cdots$. Defining $g(x) = f(x+2) - f(x+1)$, $h(x) = g(x+2) - g(x)$, and $\ell(x) = h(x+4) - h(x)$, we are interested in the value of $\ell(x)$ (this is the expression in the problem). Observe that $(x+d)^n - (x+c)^n = n(d-c)x^{n-1} + \cdots$. Hence, if $w(x) = ax^n + \cdots$, then $w(x+d) - w(x+c) = an(d-c)x^{n-1} + \cdots$. We deduce that

$$g(x) = 21x^2 + \cdots, \quad h(x) = 84x + \cdots, \quad \text{and} \quad \ell(x) = 336.$$

Thus, the answer is (e).

(30) (a) Let $y = \cos(\theta)$ and $x = \sin(\theta)$. Note that $x^2 + y^2 = 1$. DeMoivre's formula gives that $(y + \mathbf{i}x)^5 = \cos(0) + \mathbf{i}\sin(0) = 1$ for $\theta \in \{0, 2\pi/5, 4\pi/5, 6\pi/5, 8\pi/5\}$ (where θ is given in radians). Comparing the imaginary parts on both sides of the equation $(y+\mathbf{i}x)^5 = 1$, we obtain $5y^4x - 10y^2x^3 + x^5 = 0$. Since $y^2 = 1 - x^2$, we obtain $5(1-x^2)^2x - 10(1-x^2)x^3 + x^5 = 0$. Expanding, we obtain $xf(x) = 0$ where $f(x) = 16x^4 - 20x^2 + 5$. The roots of $xf(x)$ are $\sin\theta$ where $\theta \in \{0, 2\pi/5, 4\pi/5, 6\pi/5, 8\pi/5\}$. This implies the roots of $xf(x)$ are 0, $\sin(2\pi/5)$, $\sin(4\pi/5)$, $\sin(6\pi/5) = -\sin(\pi/5)$, and $\sin(8\pi/5) = -\sin(3\pi/5)$. The problem, therefore, corresponds to computing the product of the roots of $f(x)$. Using the constant term and the leading coefficient of $f(x)$, we deduce that the answer is 5/16.