Mathematics Contest
University of South Carolina
December 6, 1997

1. Evaluate the sum

$$1 - 2 + 3 - 4 + 5 - 6 + \cdots + 997 - 998 + 999 - 1000\,.$$

(a) -500 (b) -1000 (c) -999 (d) -1001 (e) 500500

2. The difference of two positive numbers is 2, and the product of these two numbers is 17. What is their sum?

(a) $6\sqrt{2}$ (b) $3\sqrt{2}-2$ (c) $6\sqrt{2}-2$ (d) $3\sqrt{2}$
(e) no such numbers exist

3. One solution of $x^3 + 5x^2 - 2x - 4 = 0$ is $x = 1$. Which of the following is another solution?

(a) $-1+\sqrt{7}$ (b) $-3+\sqrt{5}$ (c) $-2+\sqrt{5}$ (d) $-5+\sqrt{3}$
(e) $-5+\sqrt{2}$

4. At how many points does the graph of

$$y = (x-2)(2x^2-5x+4)(2x^2-7x+4)$$

intersect the x-axis?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 5

5. A bowl contains 100 pieces of colored candy: 28 green, 20 red, 12 yellow, 10 blue, 20 brown, and 10 orange. If you are blindfolded as you pick and eat candy from this bowl, then how many pieces must you eat in order to guarantee that you have eaten at least 15 of the same color?

(a) 28 (b) 40 (c) 55 (d) 75 (e) 82

6. Michael and Dave play a game in which each independently throws a dart at a target. Michael hits the target with probability 0.6, while Dave hits the target with probability 0.3. Michael wins the game if he hits the target and Dave misses. Dave wins if he hits the target and Michael misses. Otherwise the game is a tie. What is the probability that the game is a tie?

(a) 0.45 (b) 0.46 (c) 0.47 (d) 0.48 (e) 0.49

7. Which of the following has the same value as

$$\frac{1}{\log_2(100!)} + \frac{1}{\log_3(100!)} + \frac{1}{\log_4(100!)} + \cdots + \frac{1}{\log_{100}(100!)}?$$

(a) $\frac{1}{100}$ (b) 1 (c) $\frac{1}{100!}$ (d) 100

(e) $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{100}$

8. Tim's car gets 3 more miles per gallon during highway driving than it does during city driving. On a recent trip, Tim drove 136 miles on the highway and 155 miles in the city, using a total of 9 gallons of gasoline. How many miles per gallon does Tim's car get during city driving?

(a) 29 (b) 30 (c) 31 (d) 32 (e) 33

9. An equilateral triangle and a regular hexagon have equal perimeters. What is the ratio of the area of the triangle to the area of the hexagon?

(a) 1 (b) $\sqrt{3}/2$ (c) $1/2$ (d) $1/3$ (e) $2/3$

10. The perimeter of a right triangle is $12 + 8\sqrt{3}$. The sum of the squares of all three of its sides is 294. Find the area of this triangle.

(a) $11 + \sqrt{3}$ (b) $6\sqrt{3}$ (c) 12 (d) $7\sqrt{3}$ (e) $5 + 4\sqrt{3}$

11. Suppose $f(x)$ is a polynomial with integer coefficients for which 3 and 13 are both roots. Which of the following could possibly be the value of $f(10)$?

(a) 3 (b) 10 (c) 14 (d) 39 (e) 42

12. John and Nancy live on the same street and often walk towards each other's home. If they both leave their homes at 8:00 a.m., then they will meet at 8:04 a.m. If Nancy leaves her home at 8:00 a.m. but John does not leave his home until 8:03 a.m., then they will meet at 8:05 a.m. How many minutes does it take for John to walk all the way to Nancy's home? Assume that each person walks at his or her own constant rate.

(a) 8 (b) 9 (c) 10 (d) 11 (e) 12

13. The integer n is obtained by reversing the order of the digits of the 3-digit integer m. If the product of n and m is equal to 214875, then the middle digit of n must be

(a) 1 (b) 3 (c) 5 (d) 7 (e) 9

14. The quadratic polynomial $P(x)$ has the following properties: $P(x) \geq 0$ for all real numbers x, $P(1) = 0$, and $P(2) = 2$. What is the value of $P(0) + P(4)$?

(a) 19 (b) 20 (c) 21 (d) 22 (e) 23

15. A finite sequence $a_0, a_1, \ldots, a_n$ of integers is called a *curious sequence* if it has the property that for every $k = 0, 1, 2, \ldots, n$, the number of times k appears in the sequence is a_k. For example, $a_0 = 1, a_1 = 2, a_2 = 1, a_3 = 0$ forms a curious sequence. Let $a_0, a_1, \ldots, a_{100}$ be a curious sequence. What is the value of the sum $\sum_{k=0}^{100} a_k$?

(a) 99 (b) 100 (c) 101 (d) 200 (e) 201

16. The interior angles of a convex polygon of 9 sides are in arithmetic progression. If the smallest interior angle is $112°$, then what is the largest interior angle?

(a) $136°$ (b) $152°$ (c) $168°$ (d) $176°$ (e) $248°$

17. The product

$$(x+5)(x+10)(2x^2+3)(x^3+6x+16)^2(x+9)(x+4)^3(x+18)$$

is expanded. In the resulting polynomial, how many of the coefficients are odd?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

18. Find all values of a such that the quadratic equation $x^2+(a-3)x+a=0$ has two distinct positive real solutions.

(a) $a>0$ (b) $a<1$ (c) $0<a<3$ (d) $a>9$ (e) $0<a<1$

19. Consider the positive integers n having the property that 2 divides n, 3 divides $n+1$, 4 divides $n+2$, ..., 10 divides $n+8$. The first positive integer with this property is 2. Let N be the 4^{th} positive integer with this property. What is the sum of the digits of N?

(a) 12 (b) 14 (c) 16 (d) 18 (e) 20

20. If $f(x)$ satisfies $2f(x)+f(1-x)=x^2$ for all x, then $f(x)=$

(a) $\dfrac{x^2-3x+1}{2}$ (b) $\dfrac{x^2+8x-3}{9}$ (c) $\dfrac{4x^2+3x-2}{6}$

(d) $\dfrac{x^2+2x-1}{3}$ (e) $\dfrac{x^2+9x-4}{9}$

21. When written in base 8, the number n is $34112d4357$ where d denotes a digit in base 8. If n is divisible by 7, then the value of d is

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

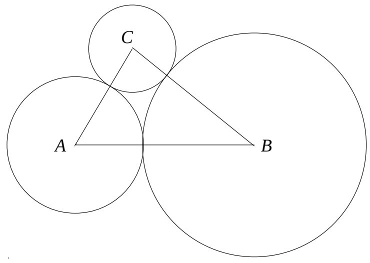
22. When $(3x + 5)^{100}$ is expanded, the largest power of 2 dividing the coefficient of x^{39} is

(a) 2^3 (b) 2^4 (c) 2^5 (d) 2^6 (e) 2^7

23. In a certain card game, 40 cards are dealt to 4 players so that each player receives 10 cards. Each card is one of 4 colors and there are 10 cards of each color numbered $1, 2, 3, \ldots, 10$. If each player receives the exact same set of 10 cards in two deals, then the two deals are considered the same. How many different deals are possible given that each player gets one card with each number?

(a) $4!^{10}$ (b) $\dfrac{40!}{10!^4}$ (c) $40!$ (d) $10!^4$ (e) 4^{10}

24. Three circles, centered at A, B, and C, are exteriorly tangent to one another. The circle with center A has radius 3. The circle with center B has radius 5. The measure of $\angle BAC$ is $\pi/3$ (in radians). What is the measure of $\angle ABC$ in radians?



(a) $\dfrac{\pi}{4}$ (b) $\arccos(11/14)$ (c) $\arccos(3/8)$ (d) $\dfrac{\pi}{6}$ (e) $\arccos(8/15)$

25. The interior of a square contains 30 points. This square, along with its interior, is partitioned into non-overlapping triangles, so that the vertices of the triangles consist of these 30 interior points together with the 4 corners of the square. If no 3 of these 34 points are collinear, then how many triangles do we obtain?

(a) 34 (b) 62 (c) 68 (d) 90 (e) 102

26. How many ways are there to choose 4 different numbers from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ so that no two of the 4 numbers are consecutive?

(a) 10 (b) 20 (c) 35 (d) 40 (e) 45

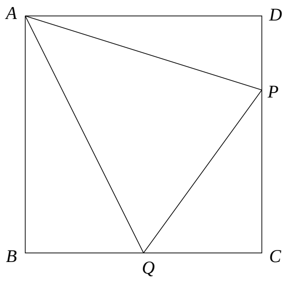
27. The Arithmetic-Geometric Mean Inequality asserts that *if* $a \geq 0$ *and* $b \geq 0$, *then* $\frac{a+b}{2} \geq \sqrt{ab}$ *and equality holds if and only if* $a = b$. Given that $x > y > 0$ and $xy = 2$, what is the smallest possible value of $\dfrac{x^2 + y^2}{x - y}$?

(a) 2 (b) $\dfrac{\sqrt{6}}{2}$ (c) $\dfrac{7}{2}$ (d) 4 (e) 5

28. Each side of the square $ABCD$ has length 1 and $\angle PAQ$ is $45°$. What is the perimeter of the triangle $\triangle PQC$?

(a) $1 + \sqrt{2}$ (b) 2 (c) $2\sqrt{2} - 1$

(d) $1 + \dfrac{1}{\sqrt{2}}$ (e) cannot be determined



29. Let $f(x) = 7x^3 + 23x + 18$. The value of

$$f(x+8) - f(x+7) - f(x+6) + f(x+5) - f(x+4) + f(x+3) + f(x+2) - f(x+1)$$

is a constant. What is its value?

(a) 7 (b) 18 (c) 42 (d) 126 (e) 336

30. What is the value of the product $\left(\sin \frac{\pi}{5}\right)\left(\sin \frac{2\pi}{5}\right)\left(\sin \frac{3\pi}{5}\right)\left(\sin \frac{4\pi}{5}\right)$?

(a) $\dfrac{5}{16}$ (b) $\dfrac{\sqrt{3}}{5}$ (c) $\dfrac{24}{55}$ (d) $\dfrac{4}{15}$ (e) $\dfrac{\sqrt{5}}{12}$