Mathematics Contest
University of South Carolina
December 7, 1996

1. Which one of the following numbers is a solution of the equation $(x^2-2)(x^2+6)(x^3-8)=0$?

   (a) 1 (b) 2 (c) 3 (d) 4 (e) 5

2. Suppose the operation $*$ is defined on the set of integers by $a*b=a+2b$. Then for every two integers a and b, the value of $a*(b*a)$ is the same as

   (a) $a*b$ (b) $b*a$ (c) $(3a)*b$ (d) $b*(4a)$ (e) $(5a)*b$

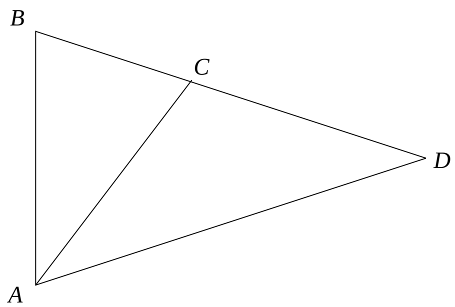
3. Given $x=20^\circ$, what is the value of $\log_5(\tan x)-\log_5(\sin x)+\log_5(\cos x)$?

   (a) 0 (b) $\frac{1}{2}$ (c) 1 (d) $\log_5 2$ (e) $\log_2 5$

4. If $\sin x+\cos x=\dfrac{1}{2}$, then what is the value of $\sin^3 x+\cos^3 x$? (Suggestion: Do not find the individual values of $\sin x$ and $\cos x$.)

   (a) $\dfrac{1}{2}$ (b) $\dfrac{3}{4}$ (c) $\dfrac{9}{16}$ (d) $\dfrac{5}{8}$ (e) $\dfrac{11}{16}$

5. Suppose that $AB=AC=CD$ and $AD=BD$. What is the measure of $\angle ADC$ in degrees?



   (a) 24 (b) 28 (c) 32 (d) 36 (e) 40

6. Which of the following numbers is the largest?

(a) 10000^{100} (b) 2^{10000} (c) 1000^{1000} (d) 5^{4000} (e) 3^{2000}

7. Given that $x^6 + 4x^5 + 6x^4 + 6x^3 + 4x^2 + 2x + 1$ can be factored as $(x^2 + ax + 1)(x^4 + bx^3 + cx^2 + dx + 1)$, what is the value of $a + b$?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

8. Suppose that $f(n) = \log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdots \log_{n-1} n$. Then the value of $\displaystyle\sum_{k=2}^{10} f(2^k)$ is

(a) 46 (b) 48 (c) 50 (d) 52 (e) 54

9. At a party, every two people shook hands once. How many people attended the party if there were 66 handshakes?

(a) 12 (b) 22 (c) 33 (d) 65 (e) 67

10. When Alice entered the Forest of Forgetfulness, she forgot the day of the week. She met the Lion and the Unicorn resting under a tree. The Lion lies on Mondays, Tuesdays and Wednesdays and tells the truth on the other days of the week. The Unicorn, on the other hand, lies on Thursdays, Fridays, and Saturdays, but tells the truth on the other days of the week. They made the following statements:

Lion: "Yesterday was one of my lying days."

Unicorn: "Yesterday was one of my lying days."

From these two statements, Alice was able to deduce the day of the week. What day was it?

(a) Monday (b) Wednesday (c) Thursday (d) Friday (e) Sunday

11. If x is a positive integer such that $x^3 + 8x^2 + 12x - 385 = 0$, then x is

(a) less than 6 (b) between 7 and 11 (c) between 12 and 20
(d) between 22 and 34 (e) larger than 50

12. Let $s(n)$ denote the sum of the digits of n. For example, $s(197) = 1 + 9 + 7 = 17$. Let $s^2(n) = s(s(n)), s^3(n) = s(s(s(n)))$, and so on. What is the value of $s^{1996}(1996)$?

(a) 18 (b) 4 (c) 12 (d) 7 (e) 0

13. Given that $f(x) = (x^5 - 1)(x^3 + 1)$, $g(x) = (x^2 - 1)(x^2 - x + 1)$, and $h(x)$ is a polynomial such that $f(x) = g(x)h(x)$, what is the value of $h(1)$?

(a) 0 (b) 2 (c) 3 (d) 5 (e) undefined

14. In a certain football league, the only way to score is to kick a field goal for 3 points or score a touchdown for 7 points. Thus the scores 1, 4 and 8 are not possible. How many positive scores are not possible?

(a) 5 (b) 6 (c) 9 (d) 11 (e) infinitely many

15. Dave can answer each problem on a certain test in 6 minutes. Michael can answer each problem in 1 minute. Suppose Michael rests for two hours in the middle of answering the problems but Dave works straight through the test without stopping. Suppose further that they finish the test at the same time. How long did it take Dave to answer all the problems?

(a) 2 hours and 10 minutes (b) 2 hours and 12 minutes
(c) 2 hours and 15 minutes (d) 2 hours and 18 minutes
(e) 2 hours and 24 minutes

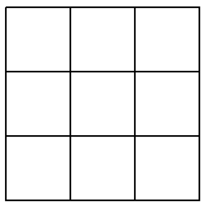
16. Suppose that $f(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$ and that $f(1) = f(2) = f(3) = f(4) = f(5)$. Then $a =$

(a) -8 (b) 10 (c) -15 (d) 22 (e) -35

17. Suppose that the angle between the minute hand and hour hand of a clock is $60°$. If the minute hand is 16 inches long and the hour hand is 10 inches long, then what is the distance between the tip ends of the hands in inches?

(a) 10 (b) 11 (c) 12 (d) 13 (e) 14

18. Suppose that the numbers 1,2,4,8,16,32,64,128,256 are placed into the nine squares in such a way that the product of the numbers appearing in any row, column or diagonal is the same. What is the value of this common product?

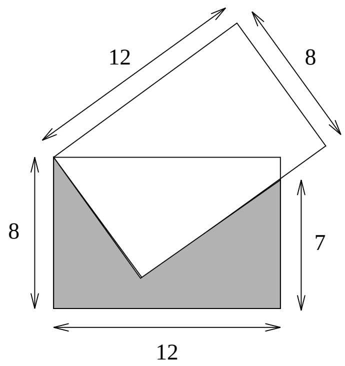


(a) 512 (b) 4096 (c) 8192 (d) 16384 (e) 32768

19. Suppose that $f(x)$ is a function such that for every real number x, (i) $f(x) + f(1 - x) = 11$ and (ii) $f(1 + x) = 3 + f(x)$. Then $f(x) + f(-x)$ must equal

(a) 8 (b) 9 (c) 10 (d) 11 (e) 12

20. Let two 8×12 rectangles share a common corner and overlap. The distance from the bottom right corner of one rectangle to the intersection point along the right edge of that rectangle is 7. What is the area of the shaded region?



(a) 36 (b) 45 (c) 48 (d) 54 (e) 56

21. A drawer contains socks. When two socks are drawn at random, the probability that both are red is 5/14. Which of the following can be the number of socks in the drawer?

(a) 5 (b) 6 (c) 7 (d) 8 (e) 9

22. Suppose that there are n points in the plane and that each of these points is joined to exactly three of the others by straight line segments. Then a possible value of n would be

(a) 11 (b) 15 (c) 18 (d) 27 (e) 39

23. We have 20 thousand dollars that must be invested among 4 possible mutual funds. Each investment must be in units of 1 thousand dollars, and there are minimal investments that need to be made if one is to invest in these funds. The minimal investments are 2, 2, 3 and 4 thousand dollars. How many different investment strategies are available if an investment must be made in each mutual fund?

(a) 220 (b) 240 (c) 260 (d) 280 (e) 300

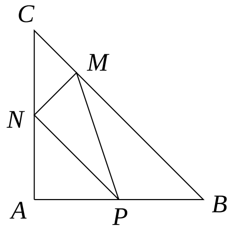
24. Ten circular coins are thrown on the street so that no two of them overlap or touch, and no three of them have a tangent line in common. What is the total number of lines which are tangent to two of the coins?

(a) 60 (b) 100 (c) 180 (d) 240 (e) 360

25. In how many ways can you walk up a stairway with 6 steps if you can take one or two steps at a time? (For example, you can walk up a stairway with 3 steps in three different ways: (i) Take three single steps. (ii) Take one step and then two steps. (iii) Take two steps and then one step.)

(a) 9 (b) 10 (c) 11 (d) 12 (e) 13

26. Suppose that $AB = AC = 4$, $\angle CAB$ is a right angle and P is the midpoint of AB. What is the smallest possible value of the perimeter of $\triangle PMN$?



(a) 5 (b) $4 + 2\sqrt{2}$ (c) 6 (d) $\sqrt{40}$ (e) $\sqrt{50}$

27. Let $f_1(x) = (x+7)^2(x^2-x+1)^3(x^5+3x^2-4)^2$. Suppose that $f_2(x) = f_1(x+1) - f_1(x)$, $f_3(x) = f_2(x+1) - f_2(x)$, $f_4(x) = f_3(x+1) - f_3(x)$, and so on. What is the smallest positive integer n such that $f_n(1) = f_n(2) = f_n(3) = \cdots = f_n(25)$?

(a) 18 (b) 19 (c) 20 (d) 22 (e) 25

28. The 46th digit after the decimal in the decimal expansion of $\frac{1}{1996}$ is

(a) 0 (b) 1 (c) 4 (d) 6 (e) 8

29. Let a and b be integers such that

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{580} + \frac{1}{581} = \frac{a}{b}.$$

If a and b have no common prime factors, what is the largest power of 3 dividing b?

(a) 3^0 (b) 3^1 (c) 3^3 (d) 3^4 (e) 3^5

30. Let $x_1, x_2, x_3, \ldots$ be the numbers which can be written as a sum of one or more different powers of 3 with $x_1 < x_2 < x_3 < \cdots$. For example, $x_1 = 3^0 = 1$, $x_2 = 3^1 = 3$, and $x_3 = 3^0 + 3^1 = 4$. What is the value of x_{100}?

(a) 972 (b) 981 (c) 999 (d) 1002 (e) 1053