Mathematics Contest
University of South Carolina
December 2, 1995

1. Suppose that $*$ is an operation on the integers defined by $a * b = a^2 + b$. What is the value of $3 * (2 * 1)$?

(a) 12 (b) 14 (c) 54 (d) 170 (e) 172

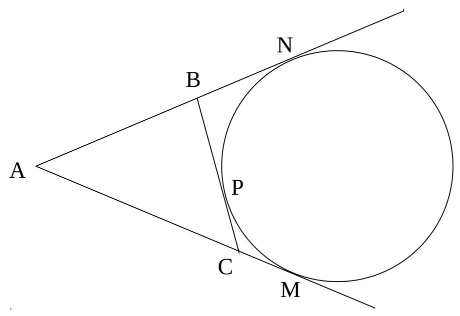
2. What is the largest prime divisor of $2^{16} - 16$?

(a) 7 (b) 11 (c) 13 (d) 17 (e) 23

3. Suppose that $f(x)$ is a function from the real numbers to the real numbers such that $f(x + f(x)) = 4f(x)$ and $f(1) = 4$. What is the value of $f(5)$?

(a) 16 (b) 18 (c) 20 (d) 22 (e) 24

4. In the figure below, the lines AN, AM and BC are tangent to the circle, and the length of AN=7. What is the perimeter of triangle ABC?



(a) 12 (b) 13 (c) 14 (d) 15 (e) 16

5. If $A + B = 12$, $B + C = 10$, and $C + D = 16$, then $A + D =$?

(a) 12 (b) 14 (c) 16 (d) 18 (e) 20

6. If $\sin x + \cos x = 1.2$, then $\sin 2x =$

   (a) 0.88 (b) 0.44 (c) -0.2 (d) -0.44 (e) -0.88

7. If r and s are the roots of $x^2 - 6x + 2 = 0$, then $\frac{1}{r} + \frac{1}{s} =$

   (a) -3 (b) 3 (c) -6 (d) 6 (e) 2

8. The average of some set of n positive numbers is 60. After removing one of the numbers, the average of the remaining n − 1 numbers is 70. What is the largest possible value of n?

   (a) 22 (b) 20 (c) 12 (d) 8 (e) 6

9. An $n \times n$ square with the entries $1, 2, \ldots, n^2$ is said to be a *magic square* if the sum of the entries of any row or column is always the same. Examples of a 3×3 magic square and a 4×4 magic square are shown below. Find the value of A in the (partially filled) 5×5 magic square.

4	3	8
9	5	1
2	7	6

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

20	13	6	4	A

   (a) 10 (b) 15 (c) 17 (d) 18 (e) 22

10. The last digit in 57^{89} is

   (a) 1 (b) 3 (c) 5 (d) 7 (e) 9

11. Suppose that x is a complex number such that $x^2 - x + 1 = 0$. What is the value of x^3?

   (a) -1 (b) 0 (c) 1 (d) 1.5 (e) 2

12. Two vertical poles of different heights stand on level ground. Straight chords from the top of each pole to the base of the other pole cross at a point 24 meters above ground. If the shorter pole is 40 meters tall, what is the height of the taller pole, in meters?

(a) 48 (b) 52 (c) 56 (d) 60 (e) 64

13. Which of the following is closest in value to $\sqrt{1.009}$?

(a) 1.002 (b) 1.0045 (c) 1.003 (d) 1.0008
(e) 1.0003

14. Suppose that $(1+x+x^3)^7 = a_0 + a_1x + a_2x^2 + \cdots + a_{20}x^{20} + a_{21}x^{21}$. What is the value of $a_3 + a_4 + a_5 + \cdots + a_{19} + a_{20} + a_{21}$?

(a) 2150 (b) 2154 (c) 2158 (d) 2162 (e) 2166

15. If $i = \sqrt{-1}$, then $(\sin 3^\circ + i \cos 3^\circ)^{60} =$

(a) -1 (b) i (c) $-i$ (d) 1 (e) none of these

16. In an infinite sequence of squares, every square except the first is formed by joining the midpoints of the sides of the previous square. What is the sum of the perimeters of the squares given that the first square has perimeter 1?

(a) $2\sqrt{2}-1$ (b) $1+2\sqrt{2}$ (c) $1+\sqrt{2}$
(d) $2+\sqrt{2}$ (e) $2(1+\sqrt{2})$

17. Atlanta is playing Cleveland in the World series. Each team has an even chance to win any given game. The World Series is won by the first team to win 4 games, so at most seven games can be played. Atlanta is leading the series 3 to 2. What is the probability that Atlanta wins the series?

(a) 0.50 (b) 0.60 (c) 0.65 (d) 0.75 (e) 0.80

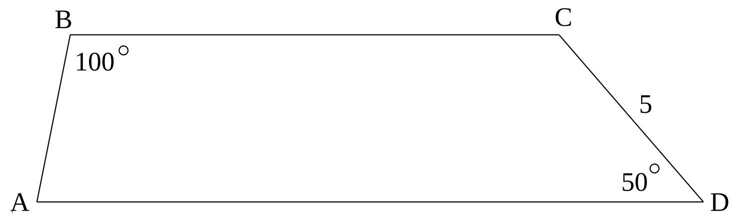
18. An old and somewhat illegible invoice shows that 72 canned hams were purchased for a total of $\$x67.9y$ (before tax), where x and y are digits. What is x?

(a) 7 (b) 6 (c) 5 (d) 4 (e) 3

19. Two jugs each have a capacity of x gallons. One is filled with wine and the other with water. Two gallons are taken from each jug and then transferred to the other, after which each jug has its contents thoroughly mixed. Next, two gallons are again taken from each jug and transferred to the other. Given that the amount of wine in each jug is now the same, find x.

(a) 3 (b) 4 (c) 5 (d) 6 (e) 7

20. The perimeter of the trapezoid below is 35. Side CD has length 5. Also, $\angle D = 50°$ and $\angle B = 100°$. What is the length of side AD?



(a) 15 (b) 16 (c) 18 (d) 20 (e) 22

21. The value of $\log_{10} 2$ is $0.301\ldots$ (accurate to as many digits as shown). How many base 10 digits does the number 5^{80} have?

(a) 56 (b) 57 (c) 58 (d) 59 (e) 60

22. Consider the following subsets of the integers: $S_1 = \{1\}$, $S_2 = \{2, 3\}$, $S_3 = \{4, 5, 6\}$, $S_4 = \{7, 8, 9, 10\}$, $S_5 = \{11, 12, 13, 14, 15\}\ldots$ where each S_n contains one more element than S_{n-1} and begins with the smallest positive integer not in any of the preceding S_k's. What is the largest element of S_{100}?

(a) 4000 (b) 4050 (c) 5000 (d) 5050 (e) 6000

23. If S is a set containing three or more integers, then there must be two integers in S whose sum is divisible by 2. For example, if $S = \{2, 11, 17\}$, then $11 + 17$ is divisible by 2. What is the smallest possible integer n such that if S is a set containing n or more integers, then there must be three integers in S whose sum is divisible by 3?

(a) 3 (b) 4 (c) 5 (d) 6 (e) no such n

24. The number of *even* positive integers that are divisors of $720 = 2^4 \times 3^2 \times 5$ is

(a) 15 (b) 16 (c) 24 (d) 25 (e) 29

25. There are 44 permutations of the numbers $1, 2, 3, 4, 5$ in which no number appears in its proper position; i.e., 1 is not first, 2 is not second, 3 is not third, etc. How many permutations of $1, 2, 3, 4, 5, 6, 7$ have exactly two numbers appearing in their proper positions (e.g., $21\underline{3}57\underline{6}4$ and $\underline{1}37\underline{4}256$)?

(a) 900 (b) 924 (c) 930 (d) 936 (e) 954

26. Suppose that the 4×4 array below is completed so that each row and each column of the final array is a permutation of the numbers $1, 2, 3, 4$. Then $a + b =$

1	2	3	4
2			1
3		a	
4	b		

(a) 2 (b) 3 (c) 4 (d) 5 (6) 6

27. The probability that Michael can win a game (any game) against Dave is 0.20. What is the smallest integer n for which the following statement is true? "If Dave and Michael play n games, then Michael has a better than 60% chance to win at least one."

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

28. Five points P_1, P_2, P_3, P_4 and P_5 are positioned in such a way that no three are collinear. Some lines are drawn with each line drawn passing through two of the five points and no two lines passing through the same two points. Suppose for $1 \leq j \leq 5$, that $l(j)$ is the number of lines drawn passing through P_j, and suppose that $\{l(1), l(2), l(3), l(4)\} = \{1, 2, 3, 4\}$. What is the value of $l(5)$?

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

29. Which of the following divide evenly into $x^{81} - x^{10} - x + 1$?

I. $x^2 + x + 1$
II. $x^2 - x + 1$
III. $x^4 + x^3 + x^2 + x + 1$
IV. $x^4 - x^3 + x^2 - x + 1$

(a) I and II only (b) III and IV only (c) I and III only
(d) II and IV only (e) II, III and IV only

30. Given that the equations below all hold for x, y, z, t and w, determine which statement (a), (b), (c), (d) or (e) is correct.

$$\begin{aligned}
x + y + z + t + w &= 1 \\
x + y + 2z + 2t + 2w &= 1 \\
x + 2y + 3z + 4t + 5w &= 2 \\
x + y + 3z + 2t + w &= 3
\end{aligned}$$

(a) x must be -2.
(b) z must be either 0 or positive.
(c) y must be negative.
(d) None of x, y, z, t and w can be uniquely determined.
(e) There are no x, y, z, t and w satisfying the equations.