MATHEMATICS CONTEST
UNIVERSITY OF SOUTH CAROLINA
4 DECEMBER 1993

INSTRUCTIONS

1. Do not open this booklet until you are told to do so.

2. This is a thirty question multiple choice test. Each question is followed by answers marked a, b, c, d, and e. Only one of these is the correct answer. Drawings on the test are not to scale.

3. Using the #2 pencil given to you, record your answers on the SCANTRON sheet (labelled with your name and school) given to you at registration. *Do not make any marks on this sheet except for your answers.* Use only the first 30 lines of the side labelled with your name. Record your answers with heavy marks, and if you must make corrections, erase thoroughly.

4. The test will be scored as follows: Five points for each correct answer, one point for an answer that is left blank, zero for an incorrect answer. Thus you are discouraged from guessing.

5. You may work the problems in this booklet using it as scratch paper. It is yours to keep after the test.

6. When you are given the signal, begin working the problems. You have 90 minutes working time for the test.

7. If your booklet has a defective, missing, or illegible page, notify the person administering the test.

8. When you have finished the test or the 90 minute time period has elapsed, return the SCANTRON sheet and pencil to the person giving the test.

Mathematics Contest
University of South Carolina
3 December 1993

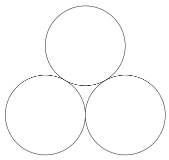
1. If the width of a particular rectangle is doubled and the length is increased by 3, then the area is tripled. What is the length of the rectangle?

(a) 1 (b) 2 (c) 3 (d) 6 (e) 9

2. Suppose the operation $\star$ is defined by $a \star b = a + b + ab$. If $3 \star x = 23$, then $x =$

(a) 2 (b) 3 (c) 4 (d) 5 (e) 6

3. If 3 circles of radius 1 are mutually tangent as shown, what is the area of the gap they enclose?



(a) $\sqrt{3} - \frac{\pi}{2}$ (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$ (e) $\frac{\pi}{6}$

4. If $(1+i)^{100}$ is expanded and written in the form $a + bi$ where a and b are real numbers, then $a =$

(a) -2^{50} (b) $20^{50} - \dfrac{100!}{50!50!}$ (c) $\dfrac{100!}{(25!)^2 50!}$

(d) $100!\left(\dfrac{-1}{50!50!} + \dfrac{1}{25!75!}\right)$ (e) 0

5. Suppose that f is a function with the property that for all x and y, $f(x+y) = f(x) + f(y) + 1$ and $f(1) = 2$. What is the value of $f(3)$?

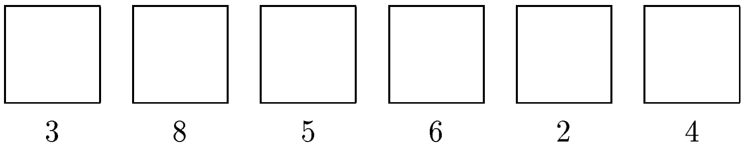
(a) 4 (b) 5 (c) 6 (d) 7 (e) 8

6. After a $p\%$ price reduction, what increase does it take to restore the original price?

(a) $p\%$ (b) $\dfrac{p}{1-p}\%$ (c) $(100-p)\%$ (d) $\dfrac{100p}{100+p}\%$

(e) $\dfrac{100p}{100-p}\%$

7. Each card below covers up a number. The number written below each card is the sum of all the numbers covered by all of the *other* cards. What is the sum of all of the hidden numbers?



(a) 4.2 (b) 5 (c) 5.6 (d) 6.2 (e) 6.8

8. What is the coefficient of x^3 in the expansion of

$$(1+x+x^2+x^3+x^4+x^5)^6?$$

(a) 40 (b) 48 (c) 56 (d) 62 (e) 64

9. Suppose that x and y are integers such that $y > x > 1$ and $y^2 - x^2 = 187$. Then one possible value of xy is

(a) 30 (b) 36 (c) 40 (d) 42 (e) 54

10. $\operatorname{Arcsin}(1/3) + \operatorname{Arccos}(1/3) + \operatorname{Arctan}(1/3) + \operatorname{Arccot}(1/3) =$

(a) π (b) $\pi/2$ (c) $\pi/3$ (d) $2\pi/3$ (e) $3\pi/4$

11. Suppose that 4 cards labeled 1 to 4 are placed randomly into 4 boxes also labeled 1 to 4, one card per box. What is the probability that no card gets placed into a box having the same label as the card?

(a) 1/3 (b) 3/8 (c) 5/12 (d) 1/2 (e) 9/16

12. If the equations (1) $x^2 + ax + b = 0$ and (2) $x^2 + cx + d = 0$ have exactly one root in common, and $abcd \neq 0$, then the other root of equation (2) is

(a) $\dfrac{c-a}{b-d}d$ (b) $\dfrac{a+c}{b+d}d$ (c) $\dfrac{b+c}{a+d}c$ (d) $\dfrac{a-c}{b-d}$ (e) $\dfrac{a+c}{b-d}c$

13. Suppose that x and y are numbers such that $\sin(x+y) = 0.3$ and $\sin(x-y) = 0.5$. Then $\sin(x)\cos(y) =$

(a) 0.1 (b) 0.3 (c) 0.4 (d) 0.5 (e) 0.8

14. How many permutations of 1, 2, 3, 4, 5, 6, 7, 8, 9 have:

1 appearing somewhere to the left of 2,

3 somewhere to the left of 4, and

5 somewhere to the left of 6?

For example, 8 1 5 7 2 3 9 4 6 would be such a permutation.

(a) $9 \cdot 7!$ (b) $8!$ (c) $5!\,4!$ (d) $8!\,4!$ (e) $8! + 6! + 4!$

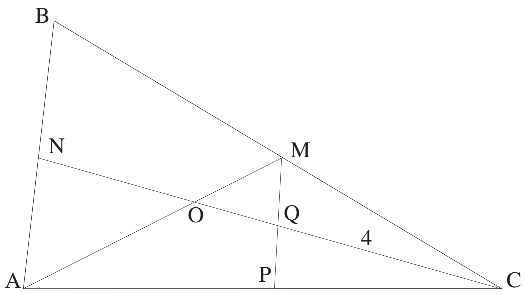
15. If we express the sum

$$\frac{1}{3\cdot 5\cdot 7\cdot 11}+\frac{1}{3\cdot 5\cdot 7\cdot 13}+\frac{1}{3\cdot 5\cdot 11\cdot 13}+\frac{1}{3\cdot 7\cdot 11\cdot 13}+\frac{1}{5\cdot 7\cdot 11\cdot 13}$$

as a rational number in reduced form, then the denominator will be

(a) 15015 (b) 5005 (c) 455 (d) 385 (e) 91

16. In the triangle below, M, N, and P are the midpoints of BC, AB, and AC respectively. CN and AM intersect at O. If the length of CQ is 4, then what is the length of OQ?



(a) 1 (b) $4/3$ (c) $\sqrt{2}$ (d) $3/2$ (e) 2

17. Let $[x]$ represent the greatest integer that is less than or equal to x. Fox example, $[2.769] = 2$ and $[\pi] = 3$. Then what is the value of

$$[\log_2 2] + [\log_2 3] + [\log_2 4] + \cdots + [\log_2 99] + [\log_2 100]?$$

(a) 480 (b) 481 (c) 482 (d) 483 (e) 484

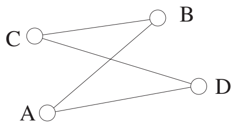
18. The minimum value of the function

$$f(x) = \frac{\sin(x)}{\sqrt{1-\cos^2(x)}} + \frac{\cos(x)}{\sqrt{1-\sin^2(x)}} + \frac{\tan(x)}{\sqrt{\sec^2(x)-1}} + \frac{\cot(x)}{\sqrt{\csc^2(x)-1}}$$

as x varies over all numbers in the largest possible domain of f, is

(a) -4 (b) -2 (c) 0 (d) 2 (e) 4

19. In the figure below, there are 4 distinct dots A, B, C, and D, joined by edges. Each dot is to be colored either red, blue, green, or yellow. No two dots joined by an edge are to be colored with the same color. How many completed colorings are possible?



(a) 24 (b) 72 (c) 84 (d) 96 (e) 108

20. Let $A_1, A_2, \ldots, A_{63}$ be the 63 nonempty subsets of $\{1,2,3,4,5,6\}$. For each of these sets A_i, let $\pi(A_i)$ denote the product of all the elements in A_i. Then what is the value of $\pi(A_1)+\pi(A_2)+\cdots+\pi(A_{63})$?

(a) 5003 (b) 5012 (c) 5039 (d) 5057 (e) 5093

21. Suppose that each pair of eight tennis players either played exactly one game last week or did not play at all. Each player participated in all but 12 games. How many games were played among the eight players?

(a) 10 (b) 12 (c) 14 (d) 16 (e) 18

22. Let

$$A = \left(1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}\right)\left(1+\tfrac{1}{3}+\tfrac{1}{9}\right)\left(1+\tfrac{1}{5}\right)\left(1+\tfrac{1}{7}\right)\left(1+\tfrac{1}{11}\right)\left(1+\tfrac{1}{13}\right),$$

$$B = \left(1-\tfrac{1}{2}\right)^{-1}\left(1-\tfrac{1}{3}\right)^{-1}\left(1-\tfrac{1}{5}\right)^{-1}\left(1-\tfrac{1}{7}\right)^{-1}\left(1-\tfrac{1}{11}\right)^{-1}\left(1-\tfrac{1}{13}\right)^{-1},$$

and

$$C = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}.$$

Then which of the following inequalities is true?

(a) $A>B>C$ (b) $B>A>C$ (c) $C>B>A$ (d) $C>A>B$
(e) $B>C>A$

23. The relation between the sets

$$M = \{12m+8n+4\ell : m,n,\ell \text{ are integers}\}$$

and

$$N = \{20p+16q+12r : p,q,r \text{ are integers}\}$$

is

(a) $M \subsetneqq N$ (b) $N \subsetneqq M$ (c) $M \cap N = \{0\}$
(d) 60244 is in M but not in N (e) $M = N$

24. If $f(x) = \dfrac{1+x}{1-3x}$, $f_1(x) = f(f(x))$, $f_2(x) = f(f_1(x))$, and in general $f_n(x) = f(f_{n-1}(x))$, then $f_{1993}(3) =$

(a) 3 (b) 1993 (c) $\dfrac{1}{2}$ (d) $\dfrac{1}{5}$ (e) -2^{-1993}

25. What is the center of the circle passing through the point $(6,0)$ and tangent to the circle $x^2+y^2=4$ at $(0,2)$? (Two circles are tangent at a point P if they intersect at P and at no other point.)

(a) $(0,-6)$ (b) $(1,-9)$ (c) $(-1,-9)$ (d) $(0,-9)$
(e) none of these

26. Let $n = 1667$. Then the first nonzero digit in the decimal expansion of $\sqrt{n^2+1}-n$ is

(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

27. Suppose $\triangle ABC$ is a triangle with area 24 and that there is a point P inside $\triangle ABC$ which is distance 2 from each of the sides of $\triangle ABC$. What is the perimeter of $\triangle ABC$?

(a) 12 (b) 24 (c) 36 (d) $12\sqrt{2}$ (e) $12\sqrt{3}$

28. Suppose $\triangle ABC$ is a triangle with 3 acute angles A, B, and C. Then the point $(\cos B - \sin A, \sin B - \cos A)$

(a) can be in the 1st quadrant and can be in the 2nd quadrant only
(b) can be in the 3rd quadrant and can be in the 4th quadrant only
(c) can be in the 2nd quadrant and can be in the 3rd quadrant only
(d) can be in the 2nd quadrant only
(e) can be in any of the 4 quadrants

29. If the sides of a triangle have lengths 2, 3, and 4, what is the radius of the circle circumscribing the triangle?

(a) 2 (b) $8/\sqrt{15}$ (c) $5/2$ (d) $\sqrt{6}$ (e) $(\sqrt{6}+1)/2$

30. $\dfrac{1}{1\cdot2\cdot3\cdot4}+\dfrac{1}{2\cdot3\cdot4\cdot5}+\dfrac{1}{3\cdot4\cdot5\cdot6}+\cdots+\dfrac{1}{28\cdot29\cdot30\cdot31}=$

(a) 1/18 (b) 1/21 (c) 4/93 (d) 128/2505 (e) 749/13485