MATERIAL TO STUDY FOR MATH 782

I have tried to list below the main topics covered in the course. If a topic is not to be included as part of the comprehensive exam, I have written "omit" next to the topic. If I expect you to know the proof(s) associated with a given topic, I have written "proof" next to the topic. (Note that this list is two pages long.)

(1) $\displaystyle\prod_{p\le x}\left(1-\frac{1}{p}\right) \le \frac{1}{\log x}$ (proof)

(2) The sum of the reciprocal of the primes diverges (proof using (1) above)

(3) Chebyshev's estimate (proof of the weaker form done in class)

(4) Abel summation

(5) Merten's formulas

(6) $\displaystyle\lim_{s\to 1^+}(s-1)\zeta(s)=1$ and $\displaystyle\lim_{s\to 1^+}(s-1)^2\zeta'(s)=-1$ where s denotes a real variable (proof using $\zeta(s)=\sum_{n=1}^{\infty}1/n^s$ rather than the analytic continuation formulation of $\zeta(s)$)

(7) Applications of the Prime Number Theorem

(8) The Prime Number Theorem (proof given

$$-\frac{\zeta'(s)}{\zeta(s)} = s\int_1^{\infty}\frac{\psi(x)}{x^{s+1}}\,dx \quad \text{for } s=\sigma+it \text{ where } \sigma>1$$

and the statement of the Wiener-Ikahara Theorem)

(9) Other aspects (lemmas) of the proof of the Prime Number Theorem from class (omit)

(10) Applications of Dirichlet's Theorem on primes in arithmetic progression

(11) Characters (in particular, computing values given certain other values of a character)

(12) Properties of characters, proofs of

$$\sum_{a\in G}\chi(a) = \begin{cases} 0 & \text{if } \chi \text{ is not principal} \\ |G| & \text{if } \chi \text{ is principal.} \end{cases}$$

and

$$\sum_{\chi}\chi(a) = \begin{cases} 0 & \text{if } a\ne e \\ |G| & \text{if } a=e \end{cases}$$

(13) Dirichlet's theorem (proof given that $w(L(s,\chi))$ remains bounded as $s \to 1^+$, see Part II of the proof of Dirichlet's theorem in the notes)

(14) Other aspects (lemmas) of the proof of the Dirichlet's Theorem from class (omit)

(15) Sieve of Eratosthenes, proof that

$$|\{n \leq x : p|n \implies p > z\}| \leq \prod_{p \leq z} \left(1 - \frac{1}{p}\right) x + 2^{\pi(z)}$$

(16) Pure Brun Sieve (omit)