COMPREHENSIVE EXAM: MATH 788F

Do as many of the problems below and on the reverse side of this page as you can. If a problem consists of several parts, you should try all the parts (you may be able to do part (c), for example, even if you cannot do part (a) or (b)).

(1) Prove that $f(x) = x^n + 216x^m - 72$ is irreducible for all positive integers n and m with $n > m$.

(2) (a) Let

$$f(x) = \left(x - \sqrt{2} - \sqrt{3}\right)\left(x - \sqrt{2} + \sqrt{3}\right)\left(x + \sqrt{2} - \sqrt{3}\right)\left(x + \sqrt{2} + \sqrt{3}\right) = x^4 - 10x^2 + 1.$$

Prove that $f(x)$ is irreducible.

(b) Prove that the polynomial $f(x)$ in part (a) is reducible modulo every prime.

(3) Prove the theorem below. You may use the indicated lemmas without proving them.

Theorem. *If $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ with $n \geq 1$ is such that $a_n = 1$, $a_0 \neq 0$, and $|a_{n-1}| > 1 + |a_{n-2}| + |a_{n-3}| + \cdots + |a_0|$, then $f(x)$ is irreducible.*

Lemma 1. *Let $f(x)$ be a monic polynomial in $\mathbb{Z}[x]$ for which $f(0) \neq 0$. Suppose further that $f(x)$ has exactly 1 root α (with multiplicity 1) such that $|\alpha| \geq 1$. Then $f(x)$ is irreducible.*

Lemma 2. *Let $f(x)$ and $g(x)$ be polynomials in $\mathbb{C}[x]$, and let $\mathcal{C} = \{z \in \mathbb{C} : |z| = 1\}$. If the strict inequality $|f(z)+g(z)| < |f(z)|+|g(z)|$ holds for each $z \in \mathcal{C}$, then $f(x)$ and $g(x)$ have the same total number of zeroes (counting multiplicity) inside the circle $\mathcal{C}$ (i.e., in the interior of the region bounded by $\mathcal{C}$).*

(4) Let $N = p_1 p_2 \ldots p_r$ where the p_j are primes which are not necessarily distinct. Suppose that the only non-empty subset S of $\{1, 2, \ldots, r\}$ for which $\prod_{j \in S} p_j \equiv 1 \pmod{10}$ is $S = \{1, 2, \ldots, r\}$. (Two examples are $N = 7 \times 13$ and $N = 17 \times 17 \times 19$.) Prove that if the decimal expansion of N is $d_n d_{n-1} \ldots d_0$, then $\sum_{j=0}^{n} d_j x^j$ is irreducible. (You can use any results we established in class related to this problem without proving them.)

(5) (a) Express $x^{100} + 1$ as the product of cyclotomic polynomials.

(b) Show that $x^{100} + 4$ factors modulo 7 as the product of 26 irreducible polynomials. (Hint: Reduce 4^{100} modulo 7.)

(c) Show that the 26 polynomials in part (b) are distinct.

(6) Let $f(x) = x^n + x^{n-1} + \cdots + x^2 + x + 1$. Prove that if $n = p - 1$ where p is an odd prime, then $f'(x)$ (the derivative of $f(x)$) is Eisenstein with respect to p. (Hint: $f'(x)$ is the derivative of $(x^{n+1} - 1)/(x - 1)$.)

(7) (a) Let $f(x) = x^n + x^{n-1} + \cdots + x^2 + x + 1$. Explain why the (complex) roots of $f'(x)$ have absolute value < 1.

(b) With $f(x)$ as in (a), show that $f'(x)$ is irreducible if $n = p^r$ where p is a prime and r is a positive integer. (Comment: The case $r = 1$ is simpler, and you may wish to do this first.)

(8) Let $f(x) \in \mathbb{Z}[x]$ with $f(x) \not\equiv 0$. Suppose that for every positive integer n, the polynomial $x^n + f(x)$ has no cyclotomic factors. Let

$$k(n) = \min\{\deg g(x) : g(x) \in \mathbb{Z}[x], g(x) \not\equiv \pm 1, g(x) \text{ divides } f(x)\}.$$

Prove that $k(n) \to \infty$ as $n \to \infty$. (Hint: I would (a) show the roots of $x^n + f(x)$ are bounded, (b) use (a) to show that the factors of $x^n + f(x)$ in $\mathbb{Z}[x]$ of degree $\leq r$ belong to a finite set S_r depending only on r and $f(x)$ but not n, (c) show that each element of S_r can divide at most one polynomial of the form $x^n + f(x)$, and (d) explain why (a), (b), and (c) imply $k(n) \to \infty$.)