NUMBER THEORY COMPREHENSIVE EXAM
FOR MATH 782 AND MATH 785

Do as many of the following problems as you can. You should not use a calculator during the exam. You should attempt all parts of all problems (since, in particular, the first part of a problem may not be the easiest part of that problem). Whenever p occurs below, it represents a prime; and (unless stated otherwise) whenever k, m, or n is used, it represents a positive integer.

(1) Prove that the Riemann zeta function $\zeta(s)$ is analytic everywhere except for a simple pole at $s = 1$ with residue 1.

(2) Consider the quadratic Gauss sum

$$G(k; n) = \sum_{r=1}^{n} e^{2\pi i k r^2/n}.$$

Show that $G(k; mn) = G(km; n)G(kn; m)$ if $\gcd(m, n) = 1$. Next prove that $G(k; p^{\alpha}) = pG(k; p^{\alpha-2}) = p^{\alpha/2}$ if α is an even integer.

(3) Prove that $\sum_{n=1}^{\infty} \dfrac{1}{p_n}$ diverges where p_n denotes the nth prime.

(4) Construct a table giving the values $\chi(n)$ of the four Dirichlet characters modulo 5. Here, $\chi \in \{\chi_1, \chi_2, \chi_3, \chi_4\}$ where χ_1 is the principal character and $n \in \{1, 2, 3, 4, 5\}$.

(5) Let p be a prime of the form $2^{2^n} + 1$ (a Fermat prime). Show that the primitive roots of p and quadratic non-residues of p are the same.

(6) For any complex x, define $B_n(x)$ by

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} z^n \qquad \text{where } |z| < 2\pi.$$

Prove that

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}.$$

Use

$$B_n = \sum_{k=0}^{n} \binom{n}{k} B_k$$

to find B_4. Using $\zeta(-n) = -B_{n+1}(1)/(n + 1)$, show that $B_{2n+1} = 0$ is equivalent to $\zeta(-2n) = 0$ for $n \geq 1$ (the trivial zeros of the Riemann zeta function).

Hint: For the first part, observe that $e^{xz} = \sum_{n=0}^{\infty} (x^n/n!) z^n$, observe the values of $B_n(0)$, and equate coefficients.

(7) Prove that

$$\zeta(s) = s \int_1^\infty \frac{[x]}{x^{s+1}}\, dx$$

and

$$s \int_1^\infty \frac{\pi(x)}{x^{s+1}}\, dx = \sum_p \frac{1}{p^s} \qquad \text{where } s = \sigma + it \text{ with } \sigma > 1.$$

(8) Prove that $(G(1, \chi))^2 = \left(\frac{-1}{p}\right) p$.

(9) Give a self-contained proof that $\log 2$ (the natural logarithm of 2) is irrational.

(10) Give a self-contained proof that e is transcendental.

(11) Prove that the set of Liouville numbers in $[0, 1]$ is uncountable.

(12) Show that

$$\int_0^{2\pi} \cos^{2n}(x)\, dx = \frac{\binom{2n}{n}}{2^{2n-1}} \pi$$

without integrating by parts.

(13) (a) Give a brief explanation as to why $\log_{10} \pi$ is irrational.

(b) Let $d_m d_{m-1} \dots d_2 d_1$ be an arbitrary string of base 10 digits (so m is a positive integer and each $d_j \in \{0, 1, \dots, 9\}$). Explain why Weyl's Theorem for uniform distribution modulo 1 implies that there exist infinitely many positive integers n such that the decimal expansion of π^n begins with the string of digits $d_m d_{m-1} \dots d_2 d_1$ (i.e., $\pi^n = d_m d_{m-1} \dots d_2 d_1 \times 10^k + E$ for some integer k and, here, $|E| < 10^k$).