# COMPREHENSIVE EXAM 08/98

Math 782 and Math 784

Do as many of the following problems as you can. You should **NOT** use a calculator during the exam. You should attempt all parts of all problems (since, in particular, the first part of a problem may not be the easiest part of that problem). You should give as many details as possible without relying on results from class; in other words, avoid using a result that is equivalent to or more general than a problem you are doing.

(1) A Dirichlet character $\chi$ modulo 68 satisfies $\chi(3) = -i$ and $\chi(35) = -1$. Calculate the values of $\chi(13)$, $\chi(26)$, and $\chi(37)$. (Note: $105 \equiv 37 \pmod{68}$.)

(2) Given that

$$-\frac{\zeta'(s)}{\zeta(s)} = s\int_1^\infty \frac{\psi(x)}{x^{s+1}}\,dx \quad \text{for } s = \sigma + it \text{ where } \sigma > 1,$$

explain how the Prime Number Theorem follows from the Wiener-Ikahara Theorem stated below. Your argument should explain why the hypotheses of the Wiener-Ikahara Theorem hold, what $A(x)$ and $f(s)$ are, and the connection with the Prime Number Theorem.

**Theorem.** *Let $A(x)$ be a non-negative, non-decreasing function of $x$, defined for $x \in [0, \infty)$. Suppose that the integral $\int_0^\infty A(x)e^{-xs}\,dx$ converges for $\sigma > 1$ to a function $f(s)$ which is analytic for $\sigma \geq 1$ except for a simple pole at $s = 1$ with residue 1. Then $\lim_{x\to\infty} e^{-x}A(x) = 1$.*

(3) For an arbitrary Dirichlet character $\chi$ modulo $m$, we defined

$$w(L(s,\chi)) = \sum_p \sum_{k=1}^\infty \frac{\chi(p^k)}{kp^{ks}} \quad \text{for } \sigma > 1.$$

For a non-principal character $\chi$, we deduced from $L(s,\chi)$ being analytic and $L(1,\chi) \neq 0$ that $w(L(s,\chi))$ remains bounded as $s \to 1^+$. Using this, prove Dirichlet's theorem that given any integers $a$ and $m$ with $m \geq 1$ and $\gcd(a,m) = 1$ there are infinitely many primes $p \equiv a \pmod{m}$.

(4) A strong version of Dirichlet's Theorem is as follows:

**Theorem.** *Let $a$ and $m$ be integers with $m \geq 1$ and $\gcd(a,m) = 1$. Let*

$$A(x) = |\{p \leq x : p \equiv a \pmod{m}\}|.$$

*Then there is a positive constant $C$ such that for all $x \geq 2$*

$$\left|A(x) - \frac{x}{\phi(m)\log x}\right| < C\frac{x}{\log^2 x}.$$

Using the above Theorem, prove that there are infinitely many primes $p$ such that both the first and final digits of $p$ are 9 (in other words, $p = \sum_{j=0}^r d_j 10^j$ where $d_j \in \{0, 1, 2, \ldots, 9\}$ for each $j$ and $d_r = d_0 = 9$).

(5) Let $A$ be a set of positive integers, and define $A(x)$ as the number of elements of $A$ which are $\leq x$. Suppose that $A(x) \sim x/\sqrt{\log x}$.

(a) Prove that $\displaystyle\sum_{\substack{n \leq x \\ n \in A}} \frac{1}{n} \sim c\sqrt{\log x}$ for some constant $c$ and determine the value of $c$.

(b) Explain why such a set $A$ exists or prove that it does not exist. (Hint: This part is independent of part (a).)

(6) Let $\alpha$ be a root of the irreducible polynomial $f(x) = x^3 - 3x^2 - 4$ (you do not need to explain why $f(x)$ is irreducible; this is given information).

(a) What is the minimal polynomial of $1/\alpha$?

(b) Compute $\mathrm{N}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^3)$ and $\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha^3)$.

(c) Compute $\Delta(1, \alpha, \alpha^2)$.

(7) Prove the following theorem from class:

**Theorem.** *Let $\alpha \in \mathbb{R}$. For every two consecutive convergents of the simple continued fraction for $\alpha$, one of the convergents, say $a/b$ with $\gcd(a, b) = 1$, satisfies*

$$\left|\alpha - \frac{a}{b}\right| \leq \frac{1}{2b^2}.$$

(8) Prove the following theorem from class:

**Theorem.** *Let $R$ be the ring of algebraic integers in $\mathbb{Q}(\sqrt{N})$. Then $R$ is a Euclidean domain for $N = -1, -2, -3, -7$, and $-11$.*

(9) (a) It is a fact that if $p \equiv 1 \pmod{28}$, then $-7$ is a square modulo $p$. This can be shown by evaluating the Legendre symbol $\left(\dfrac{-7}{p}\right)$, but you do not need to justify this. Use this fact to prove that if $p \equiv 1 \pmod{28}$, then there are unique positive integers $x$ and $y$ such that $p = x^2 + 7y^2$.

(b) Let $n$ be an integer such that (i) $n \equiv 1 \pmod{28}$ and (ii) there are positive integers $x$ and $y$ satisfying $n = x^2 + 7y^2$. Must $n$ be a prime? In other words, either prove that $n$ is a prime or show that $n$ may be composite.

(10) (a) Let $a$, $b$, and $c$ be arbitrary positive integers. Explain why the Thué-Siegel-Roth Theorem implies that there are at most finitely many pairs $(x, y)$ with $x$ and $y$ positive integers satisfying $ax^3 - by^3 = c$. (Hint: Use that $x^3 - (b/a)y^3 = c/a$ and factor $x - \sqrt[3]{b/a}\,y$ from the left-hand side.)

(b) Explain why (a) implies that there are only finitely many positive integers $k$ and $\ell$ such that $36^k - 35^\ell = 1$. (Hint: $k$ and $\ell$ must be somewhat close to multiples of 3.)