COMPREHENSIVE EXAM FOR MATH 782 & 784

(1) A set S is said to be dense in $\mathbb{R}$ if for every $t \in \mathbb{R}$ and every $\varepsilon > 0$, there is an element of S in $(t, t+\varepsilon)$. It is well-known that the set of rational numbers is dense in $\mathbb{R}$. Prove that in fact the set

$$T = \{\pm p/q : p \text{ and } q \text{ prime}\}$$

is dense in $\mathbb{R}$.

(2) Let S be the set of positive integers n of the form $n = pq$ where p and q are distinct primes, and let $S(x) = |\{n \le x : n \in S\}|$.

(a) Explain why $S(x) = \displaystyle\sum_{q \le \sqrt{x}} \sum_{q < p \le x/q} 1$. (Here p and q denote *primes.*)

(b) Prove that $\dfrac{x \log\log x}{\log x} \ll S(x) \ll \dfrac{x \log\log x}{\log x}$.

(3) Let S be the set given in problem (2).

(a) Using $\sum_{p \le z} 1/p = \log\log z + O(1)$ (and not the Prime Number Theorem) prove that

$$(\log\log x)^2 \ll \sum_{n \in S, n \le x} \frac{1}{n} \ll (\log\log x)^2.$$

(b) Using Abel summation and the estimate for $S(x)$ in (2)(b), prove the inequalities in (3)(a).

(c) Let S_k denote the set of positive integers n which have exactly k prime factors all of which are distinct. Thus, $S_2 = S$. Let $S_k(x) = |\{n \le x : n \in S_k\}|$. It is known that

$$S_k(x) \sim \frac{x(\log\log x)^{k-1}}{(k-1)!\,\log x}.$$

Using this information, prove that

$$\sum_{n \in S_k, n \le x} \frac{1}{n} \sim c_k (\log\log x)^k$$

for some constant c_k (depending on k). Also, determine the value of c_k.

(4) Given that

$$-\frac{\zeta'(s)}{\zeta(s)} = s \int_1^\infty \frac{\psi(x)}{x^{s+1}}\,dx \quad \text{for } s = \sigma + it \text{ where } \sigma > 1,$$

explain how the Prime Number Theorem follows from the Wiener-Ikahara Theorem stated below. Your argument should explain why the hypotheses of the Wiener-Ikahara Theorem hold, what $A(x)$ and $f(s)$ are, and the connection with the Prime Number Theorem.

Theorem. *Let $A(x)$ be a non-negative, non-decreasing function of x, defined for $x \in [0,\infty)$. Suppose that the integral $\int_0^\infty A(x) e^{-xs}\,dx$ converges for $\sigma > 1$ to a function $f(s)$ which is analytic for $\sigma \ge 1$ except for a simple pole at $s = 1$ with residue 1. Then $\lim_{x\to\infty} e^{-x} A(x) = 1$.*

(5) For an arbitrary Dirichlet character χ modulo m, we defined

$$w(L(s,\chi)) = \sum_p \sum_{k=1}^\infty \frac{\chi(p^k)}{kp^{ks}} \quad \text{for } \sigma > 1.$$

For a non-principal character χ, we deduced from $L(s,\chi)$ being analytic and $L(1,\chi) \neq 0$ that $w(L(s,\chi))$ remains bounded as $s \to 1^+$. Using this, prove Dirichlet's theorem that given any integers a and m with $m \geq 1$ and $\gcd(a,m) = 1$ there are infinitely many primes $p \equiv a \pmod{m}$.

(6) Let α be a root of the irreducible polynomial $f(x) = x^8 - 2x^6 - 2$ (you do not need to explain why $f(x)$ is irreducible; this is given information). Let β be a root of $g(x) = x^4 - 2x^3 - 2$.
(a) Why is $g(x)$ the minimal polynomial of β?
(b) Must β be in $\mathbb{Q}(\alpha)$? Explain your answer.
(c) Suppose $\beta \in \mathbb{Q}(\alpha)$. Must β be in the ring of algebraic integers in $\mathbb{Q}(\alpha)$? Explain (briefly).
(d) Suppose $\beta \in \mathbb{Q}(\alpha)$. Compute $\mathrm{N}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\beta)$ and $\mathrm{Tr}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\beta)$.
(e) Compute $\Delta(1, \alpha, \alpha^2, \ldots, \alpha^7)$.

(7) Prove that the ring R of algebraic integers in $\mathbb{Q}(\sqrt{-7})$ is a Euclidean domain. (This is part of a theorem from class.)

(8) The equation $4y^2 + 7 = 11x^3$ has the obvious solutions $(x,y) = (1, \pm 1)$. The purpose of this problem is to prove that there are finitely many solutions in integers x and y.
(a) Let R be the ring of algebraic integers in $\mathbb{Q}(\sqrt{-7})$. Is 11 a prime in R? Explain.
(b) Let $f(x) = x^3 + 6x^2 - 21x - 14$. Show that the roots of $f(x)$ are distinct.
(c) Explain why $b^3 f(a/b) = 8$ has finitely many solutions in integer pairs (a,b). (Hint: Use the Thue-Siegel-Roth Theorem.)
(d) Prove that there are finitely many integer pairs x and y satisfying $4y^2 + 7 = 11x^3$. (Hint: Consider the argument we gave for finding the integer solutions to $y^2 + 4 = x^3$.)

(9) Let $\alpha \in \mathbb{R} - \mathbb{Q}$. We know that one of every two consecutive convergents of the simple continued fraction for α is some a/b, with a and b relatively prime integers and $b > 0$, such that $|\alpha - (a/b)| < 1/(2b^2)$. Let q_j and a_j/b_j denote the partial quotients and convergents of the simple continued fraction for α, respectively.
(a) Prove that if $q_{n+1} \geq 2$ for some $n \geq 0$, then $|\alpha - (a_n/b_n)| < 1/(2b_n^2)$.
(b) Prove that for each $n \geq 0$, at least one of b_n and b_{n+1} is odd.
(c) Prove that there exist infinitely many integer pairs (a,b) with $b > 0$ *and b odd* such that $|\alpha - (a/b)| < 1/(2b^2)$. (Hint: Suppose $|\alpha - (a_{n+1}/b_{n+1})| < 1/(2b_{n+1}^2)$ but b_{n+1} is even. Consider $b_{n+1} = b_n q_{n+1} + b_{n-1}$ and the parity of b_n and b_{n-1}.)

(10) Let B denote a positive integer. This problem concerns solutions to the equation $x^2 - 15y^2 = B$.
(a) Suppose x and y satisfy $x^2 - 15y^2 = B$. Let $u = 4x - 15y$ and $v = 4y - x$. Prove that $u^2 - 15v^2 = B$ by direct substitution (express $u^2 - 15v^2$ in terms of x and y).
(b) Compute the fundamental unit in R, the ring of algebraic integers in $\mathbb{Q}(\sqrt{15})$. Show how the formulas for u and v in part (a) can be obtained by making use of this fundamental unit in R. (Think in terms of the norms of $x + \sqrt{15}y$ and the fundamental unit.)
(c) Prove that if x and y are positive in part (a), then $x/y > \sqrt{15}$.
(d) Suppose B is a positive integer for which $x^2 - 15y^2 = B$ has a solution in integers x and y. Prove that such a solution exists with

$$0 \leq y < \left(\frac{B}{2\sqrt{15}} \right)^{1/2}.$$

(Hint: When is $|v| < y$?)
(e) Suppose B is a positive integer ≤ 100. Find the set $S \subseteq \{1, 2, \ldots, 100\}$ for which the following holds: $x^2 - 15y^2 = B$ has a solution in integers x and y if and only if $B \in S$. Justify your answer.