COMPREHENSIVE EXAM 08/98

Math 780 and Math 788T

Do as many of the following problems as you can. You should **NOT** use a calculator during the exam. You should attempt all parts of all problems (since, in particular, the first part of a problem may not be the easiest part of that problem). You should give as many details as possible without relying on results from class; in other words, avoid using a result that is equivalent to or more general than a problem you are doing.

(1) (a) Prove Euler's Theorem that for every positive integer n and every integer a relatively prime to n, we have $a^{\phi(n)} \equiv 1 \pmod{n}$.

(b) Prove or disprove that for every positive integer n and every integer a (not necessarily relatively prime to n), we have $a^{\phi(n)+1} \equiv a \pmod{n}$.

(2) We say that p, $p+2$, and $p+6$ form a "prime triple" if each of these three numbers is a prime. For example, when $p = 5$, we get the prime triple 5, 7, and 11. Prove that if p, $p+2$, and $p+6$ form a prime triple with $p > 5$, then either $p \equiv 11 \pmod{30}$ or $p \equiv 17 \pmod{30}$.

(3) Let $N = 1998$ and, for each positive integer n, define

$$a_n = 1 + 2^N + 2^{2N} + \cdots + 2^{(n-1)N}.$$

(a) Prove that if p is a sufficiently large prime, then there are infinitely many positive integers n such that $p | a_n$.

(b) Determine all primes p for which there do not exist infinitely many positive integers n such that $p | a_n$.

(4) Using properties of the Legendre symbol, Dirichlet's Theorem on primes in arithmetic progression, and the Chinese Remainder Theorem, explain why for each squarefree integer $k > 1$, there are infinitely many primes p for which $\left(\frac{k}{p}\right) = 1$ and infinitely many primes q for which $\left(\frac{k}{q}\right) = -1$.

(5) Recall that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Prove that the density of squarefree numbers is $6/\pi^2$ (in other words, prove that the number of squarefree numbers $\leq x$ is asymptotic to $(6/\pi^2)x$).

(6) (a) Prove that for every real number $x > 1$ the interval $[x - 4x^{1/3}, x]$ contains a number which can be represented as $a^2 + b^3$ with a and b positive integers.

(b) Prove that there exist infinitely many positive integers x such that the interval $[x - 0.5x^{1/6}, x]$ does not contain a number which can be represented as $a^2 + b^3$ with a and b positive integers.

(7) Let $\frac{a}{b} < \frac{2}{3} < \frac{c}{d}$ be three **consecutive** elements of the Farey sequence $\mathcal{F}_n$.

(a) Find a, b, c and d when $n = 593$.

(b) Find a, b, c and d when $n = 3k + 2$ with k a positive integer.

(8) Let $f : [a, b] \mapsto \mathbb{R}$ be a function with continuos n-th derivative on the interval $[a, b]$. Let $0 < \delta < \lambda$. Assume that $\lambda < f^{(n)}(x) < 2\lambda$ for every $x \in [a, b]$. Suppose that $S = \{(x_1, y_1), (x_2, y_2), \dots, (x_k, y_k)\}$ is a set of integer points such that $|y_i - f(x_i)| < \delta$ for $i = 1, 2, \dots, k$.

(a) Prove that

$$x_{i+n} - x_i > 0.5\lambda^{-\frac{2}{n(n+1)}}$$

for $i = 1, \dots, k - n$.

(b) Prove that

$$|S| \le 2n(b - a)\lambda^{\frac{2}{n(n+1)}} + n.$$

(9) We say that a positive integer n is cubefull if any prime divisor of n occurs to the third or higher power (in other words, if p is prime and p divides n then p^3 divides n). Prove that each cubefull number can be represented as $a^3b^4c^5$ where a, b, and c are positive integers and the product bc is squarefree.

(10) Let x be a real positive number, and let $S(x)$ be the number of integers in the interval $(x, x + x^{1/8}]$ which can be written as a^3b^5 with a and b positive integers and $x^{1/8} < a < 2x^{1/8}$. Determine the smallest value of θ, obtainable from the theorems on the extra sheet provided with this test, such that for every $\epsilon > 0$, the formula $S(x) \ll x^{\theta+\epsilon}$ holds provided x is sufficiently large (i.e., provided $x \ge x_0(\epsilon)$ where $x_0(\epsilon)$ is a constant depending only on ϵ).