Comprehensive Exam: Math 780 & 788T

(1) Let $N = 2^3 \times 3^2 \times 7 \times 17$. Find the smallest integer $m > 3$ such that $a^m \equiv a^3 \pmod{N}$ for every integer a? Justify your answer.

(2) Find a positive integer k such that $k \times 11 \times 12^n + 1$ is composite for all nonnegative integers n. (Hint: Try making use of the primes 5, 13, and 29.)

(3) (a) Let a and b be relatively prime integers, and let q be a prime that is $\equiv 3 \pmod{4}$. Show that if p is an odd prime dividing $a^2 + qb^2$, then p is a square modulo q.

(b) Show that if q is a prime that is $\equiv 3 \pmod{4}$ and p is an odd prime that is a square modulo q, then there are relatively prime integers a and b such that p divides $a^2 + qb^2$.

(4) (a) Let k be a positive integer, and let p be a prime $> k$. Show that p divides $k! + 1$ if and only if $(-1)^{k+1}(p - k - 1)! + 1$ is divisible by p.

(b) Find the smallest positive integer $n > 1$ that divides $63! + 1$. (Hint: The answer is < 80.)

(5) Write down a proof that the sum of the reciprocals of the primes diverges. Use any results you think appropriate for this problem (for example, you can give an "elementary" proof or a proof based on the Prime Number Theorem and Riemann-Stieltjes integrals). Part of your grade will be based on what you view as appropriate for this problem.

(6) (a) Prove that for every real number $x > 1$ the interval $[x - 7x^{4/9}, x]$ contains a sum of two cubes of non-negative integers.

(b) Prove that for every integer $n > 125000$ the interval $[(n-1)^2, n^2]$ contains a sum of two cubes of non-negative integers.

(c) Find a positive integer n such that the interval $[(n-1)^2, n^2]$ does not contain a sum of two cubes of non-negative integers.

(7) Let $\frac{a}{b} < \frac{c}{d}$ be two **consecutive** elements of the Farey sequence $\mathcal{F}_n$.

(a) Prove that $\frac{1}{n} \geq \frac{c}{d} - \frac{a}{b} \geq \frac{1}{n^2}$.

(b) Let δ be a fixed real number such that $\frac{1}{n} > \delta > \frac{1}{n^2}$. Suppose that $\frac{c}{d} - \frac{a}{b} > \delta$. Prove that $\min(b, d) < \delta^{-\frac{1}{2}}$.

(c) Prove that the number of elements $\frac{a}{b}$ of the Farey sequence $\mathcal{F}_n$ that have the property that *the distance from* $\frac{a}{b}$ *to its right neighbor* $\frac{c}{d}$ *is greater than* δ does not exceed $\frac{2}{\delta}$.

(8) Let $n > 1$ be an integer. Denote by $A(n)$ the number of squarefree integers in the interval $[1, n]$, and by $B(n)$ the number of squarefull integers in the interval $[1, n]$. Note that 1 is squarefree but not squarefull.

(a) Prove that $A(n) > n(1 - \sum_p \frac{1}{p^2})$ where the summation is over all primes p.

(b) Prove that $A(n) > n/2$. You can use that $\displaystyle\sum_p \frac{1}{p^2} < \frac{1}{4} + \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2} = \frac{1}{4} + \frac{\pi^2}{8} - 1.$

(c) Prove that $B(n) < \displaystyle\sum_{1 \le b \le n^{\frac{1}{3}}} \left[\sqrt{\frac{n}{b^3}}\right] < \sqrt{n} \sum_{b=1}^{\infty} \frac{1}{b^{\frac{3}{2}}}$. You can use that every squarefull number m can be represented in a unique way as $m = a^2 b^3$ with a a positive integer and b a positive squarefree integer.

(d) Prove that $B(n) < 3\sqrt{n}$.

(e) Prove that $A(n) > 2B(n)$ for $n > 36$. Prove that $A(9) = 2B(9)$. What is $B(36)$? Prove that $A(n) \ge 2B(n)$ for all integers $n \ge 1$.

(9) Let x be a real positive number, and let $S(x)$ be the number of integers in the interval $(x, x + x^{5/8}]$ which can be written as $a^2 b^3$ with a and b positive integers and $x^{1/5} < a < 2x^{1/5}$. Determine the smallest value of θ, obtainable from the theorems below, such that for every $\epsilon > 0$, the formula $S(x) \ll x^{\theta+\epsilon}$ holds provided x is sufficiently large (i.e., provided $x \ge x_0(\epsilon)$ where $x_0(\epsilon)$ is a constant depending only on ϵ).

Some Math788T Theorems

Theorem 1: Let $f : [a, b] \mapsto R$ be a function with continuous second derivative on $[a, b]$. Let $0 < \lambda \le f''(x) \le c\lambda$ for all $x \in [a, b]$, and let $4\delta^2 < \lambda$. Let $S = \{(u, v) : u, v$ are integers, $u \in [a, b]$ and $|v - f(u)| < \delta\}$. Then S is a convex set. Also, $S = S_1 \bigcup S_2$ where S_1 is a strictly convex set with

$$|S_1| \le 4(c\lambda)^{1/3}(b - a) + 2$$

and

$$|S_2| \le 64 + 64c\lambda(b - a) + 8(\delta/\lambda)^{1/2}.$$

Theorem 2: Let $f \in C^3[M, 2M]$ and let $\delta < 1/32$. Let $S \subset \{u \in [M, 2M] \bigcap Z : ||f(u)|| < \delta\}$. Assume that the set $\{(u, ||f(u)||) : u \in S\}$ is a strictly convex set. Let $\frac{T}{c_2 M^2} \le |f''(x)| \le \frac{c_2 T}{M^2}$ and $\frac{T}{c_3 M^3} \le |f'''(x)| \le \frac{c_3 T}{M^3}$ for all $x \in [M, 2M]$, and let $T \ge 1$ and $M \ge 1$. Then

$$|S| \ll T^{1/6} M^{1/2} + T^{1/4} M^{1/2} \delta^{1/4} + M\delta.$$

Theorem 3: Let $f \in C^3[M, 2M]$, and let $\Delta \le |f''(x)| \le c\Delta$ and $\frac{\Delta}{c_2 M} \le |f'''(x)| \le \frac{c_2 \Delta}{M}$ for all $x \in [M, 2M]$. Assume that $\Delta < 1$. Let $0 < \delta \le 1/2$ and $4\delta^2 \le \Delta$. Define $S = \{u \in [M, 2M] \bigcap Z : ||f(u)|| < \delta\}$. Then

$$|S| \ll \Delta^{3/10} M^{9/10} \log^{2/5} M + \Delta^{1/5} M^{4/5} \log^{2/5} M + \Delta^{4/11} M^{10/11} \log^{4/11} M +$$

$$\delta^{1/8} \Delta^{3/8} M \log^{1/2} M + \delta^{1/7} \Delta^{1/7} M^{6/7} \log^{4/7} M + \delta^{2/5} \Delta^{1/5} M \log^{4/5} M + \delta M + \sqrt{\frac{\delta}{\Delta}} + 1.$$