SOLUTIONS TO USC'S 2004 HIGH SCHOOL MATH CONTEST

1. **(d)** Observe that BC is the height of $\triangle ABD$ with base $\overline{AD}$. Hence, the area of $\triangle ABD$ is $(1/2)\cdot 3\cdot 6 = 9$.

2. **(e)** Square both sides of the equation $\sqrt{3-x}+\sqrt{3+x} = x$ to obtain $6+2\sqrt{9-x^2} = x^2$. It is easy to see that $x = 2\sqrt{2} = \sqrt{8}$ is a solution. To solve for x, one can square both sides of $2\sqrt{9-x^2} = x^2-6$ and simplify to obtain $x^4-8x^2 = 0$. Clearly, $x = 0$ is not a solution to the original equation. We deduce $x = \pm 2\sqrt{2}$ are the only possible solutions. It is not difficult to check that $x = 2\sqrt{2}$ and not $x = -2\sqrt{2}$ leads to a solution using
$$3-2\sqrt{2} = (\sqrt{2}-1)^2 \quad \text{and} \quad 3+2\sqrt{2} = (\sqrt{2}+1)^2.$$

3. **(a)** By the Pythagorean Theorem, $BD = 10$. Let E be a point on $\overline{AB}$ with $\angle DEA = 90^\circ$. Then $DE = 8$ and $AE = 21-6 = 15$. By the Pythagorean Theorem again, $AD = 17$. Hence, the answer is $10+17 = 27$.

4. **(b)** The answer follows from the change of base formula for logarithms. We deduce
$$(\log_2 3)(\log_3 5)(\log_5 8) = (\log_2 3)\cdot\frac{\log_2 5}{\log_2 3}\cdot\frac{\log_2 8}{\log_2 5} = \log_2 8 = 3.$$

5. **(e)** From $\sin x = 2\cos x$ and $\sin^2 x+\cos^2 x = 1$, we deduce that $\cos^2 x = 1/5$ and, consequently, $\sin^2 x = 4\cos^2 x = 4/5$. Thus, $(\sin^2 x)(\cos^2 x) = 4/25$. It follows that $\sin x\cos x = \pm 2/5$. On the other hand, $\sin x = 2\cos x$ implies that $\sin x$ and $\cos x$ cannot have opposite signs. Hence, $\sin x\cos x = 2/5$. Alternatively, one can observe that the given information implies $\tan x = 2$ so that $\sin x = \pm 2/\sqrt{5}$ and $\cos x = \pm 1/\sqrt{5}$ (both with the same sign). The answer follows.

6. **(d)** Since $f(x) = ax+b$, we have $f(f(x)) = a(ax+b)+b = a^2x+b(a+1)$ and $f(f(f(x))) = a\big(a^2x+b(a+1)\big)+b = a^3x+b(a^2+a+1)$. Since this must equal $8x+21$, we deduce that $a = 2$ and $b = 21/7 = 3$. Hence, $a+b = 5$.

7. **(b)** It is well-known that $CB^2 = CD\cdot CA$. Also, as $\triangle ACB$ is a 30°-60°-90° triangle, we have $CA = 2CB$. Thus, $CB^2 = 2\cdot CD\cdot CB$ which implies $CB = 2\cdot CD = 2\sqrt{3}$. Again, using that $\triangle ACB$ is a 30°-60°-90° triangle, we obtain $AB = \sqrt{3}\cdot CB = 6$.

8. **(c)** The probability that the random coin is a fair coin and that it comes up heads is $0.4\cdot 0.5 = 0.2$. The probability that the random coin is a biased coin and that it comes up heads is $0.6\cdot 0.8 = 0.48$. Thus, the probability that the random coin comes up heads is $0.2+0.48 = 0.68$.

9. **(a)** The answer follows from
$$\big(\sqrt{2}\big)^{\log_2 9} = \big(\sqrt{2}\big)^{2\log_2 3} = 2^{\log_2 3} = 3.$$

10. **(a)** From $2ab = (a+b)^2-(a^2+b^2) = 4-5 = -1$, we obtain that $ab = -1/2$. Thus,
$$a^3+b^3 = (a+b)(a^2-ab+b^2) = 2\cdot(5+0.5) = 11.$$
Alternatively, one can use $a+b = 2$ and $ab = -1/2$ to deduce that a and b are $(2+\sqrt{6})/2$ and $(2-\sqrt{6})/2$ in some order. Thus, a^3+b^3 can be computed directly.

11. **(b)** Let $f(x) = x^{11}+x^{10}+\cdots+x+1$. Set $g(x) = (x-1)f(x) = x^{12}-1$. If α is a root of $f(x)$, then α is a root of $g(x)$. The only real 12^{th} roots of 1 are 1 and -1. Note that 1 is clearly not a root of $f(x)$ and -1 is a root of $f(x)$. Hence, $f(x)$ has exactly one real root.

12. **(c)** Let r denote the radius of the circle. Then the circumscribing square will have side length $2r$. The inscribed square will have diagonal $2r$ and, hence, side length $\sqrt{2}r$. The ratio is therefore $(2r)^2/(\sqrt{2}r)^2 = 2$.

13. **(e)** Take the sum of the first and third equations and subtract 3 times the second to get $e = 3$.

14. **(d)** Multiplying the given equation by x, one obtains $x^2 - 2\cos(12^\circ)x + 1 = 0$. The quadratic formula and $\cos^2(12^\circ) - 1 = -\sin^2(12^\circ)$ imply that x is one of $a = \cos(12^\circ) + \mathrm{i}\sin(12^\circ)$ or $b = \cos(12^\circ) - \mathrm{i}\sin(12^\circ)$. Since $ab = 1$, we get that if $x + (1/x) = 2\cos(12^\circ)$, then $x^5 + (1/x^5) = a^5 + b^5$. Recall that $e^{\mathrm{i}\theta} = \cos\theta + \mathrm{i}\sin\theta$ where θ is in radians and $e^{-\mathrm{i}\theta} = \cos\theta - \mathrm{i}\sin\theta$ (the latter follows from the former). As $2\pi/30$ radians is the same as 12°, we deduce that

$$x^5 + \frac{1}{x^5} = a^5 + b^5 = \left(e^{\mathrm{i}2\pi/30}\right)^5 + \left(e^{-\mathrm{i}2\pi/30}\right)^5 = e^{\mathrm{i}\pi/3} + e^{-\mathrm{i}\pi/3} = 2\cos(\pi/3) = 1.$$

Alternatively, one can show that $x^k + (1/x^k) = 2\cos(k \cdot 12^\circ)$ for every positive integer k by induction on k. Then setting $k = 5$, one obtains the same answer. We omit the details for this alternative argument.

15. **(e)** Multiplying through by $x + 1$, we see that the roots are the same as the roots of

$$(x^2 + 1)(x^4 + 1)(x^6 + 1) - (x^2 - 1) = 0$$

(where we have used that $x = -1$ is not a root of the equation above). Clearly, if a is a root, then so is $-a$. It follows that the sum of the roots is 0.

16. **(a)** Let q be the quotient and r the remainder when n is divided by 7. Then q and r are integers, with $0 \le r < 7$, satisfying $n = 7q + r$. With $n = 7q + r$, we see that there are integers k and ℓ such that

$$n^6 = 7k + r^6 \quad \text{and} \quad n^3 = 7\ell + r^3.$$

It follows that $4(n^6 - r^6) + (n^3 - r^3)$ is divisible by 7. We deduce that $4n^6 + n^3 + 5$ is divisible by 7 precisely when

$$4n^6 + n^3 + 5 - \left(4(n^6 - r^6) + (n^3 - r^3)\right) = 4r^6 + r^3 + 5$$

is. One checks directly that for $0 \le r < 7$, the number $4r^6 + r^3 + 5$ is not divisible by 7. So the answer is 0. The argument can be simplified using modulo arithmetic and, for example, noticing that $r^3 \equiv 0, 1, \text{ or } 6 \pmod 7$.

17. **(e)** Any 5 numbers chosen from the set $\{1, 2, \dots, 9\}$ determine exactly one sequence of a_j as indicated. So the answer is the same as the number of ways of choosing 5 numbers from the 9 numbers in $\{1, 2, \dots, 9\}$. The answer is $\binom{9}{5} = 9 \cdot 8 \cdot 7 \cdot 6/4! = 126$.

18. **(d)** The second equation plus twice the first equation gives

$$-1 = 17 + (-9 \cdot 2) = x^2 + 2xy + y^2 + 2x + 2y = (x + y)^2 + 2(x + y).$$

Adding 1 to both sides, we obtain $0 = (x + y + 1)^2$ so that $x + y = -1$. From the first equation, we deduce $xy = -8$. From

$$(z - x)(z - y) = z^2 - (x + y)z + xy = z^2 + z - 8,$$

we see that x and y are roots of $z^2 + z - 8$. Since $z^2 + z - 8$ has the two roots $u = (-1 + \sqrt{33})/2$ and $v = (-1 - \sqrt{33})/2$, there are two possibilities for (x, y), namely (u, v) and (v, u). One checks directly that these are solutions, so the answer is 2.

19. **(c)** From the given, we have

$$\begin{aligned} 5a^2 = a^2 + (2a)^2 &= (\sin x + \sin y)^2 + (\cos x + \cos y)^2 \\ &= (\sin^2 x + \cos^2 x) + (\sin^2 y + \cos^2 y) + 2(\sin x \sin y + \cos x \cos y) \\ &= 1 + 1 + 2\cos(x - y). \end{aligned}$$

Hence, $\cos(x - y) = (5a^2 - 2)/2$.

20. **(b)** There are 7! ways of arranging seven people in a row. If A and B are the two people who initially sat on the aisle, then after intermission A can sit in any of the 5 non-aisle seats and then B can sit in any of the remaining 4 non-aisle seats. There are five other seats, and the remaining five people can sit in these seats in $5!$ ways. Thus, the probability is $(5 \cdot 4 \cdot 5!)/7! = 10/21$.

21. **(b)** Let O denote the center of the circle, G the intersection of $\overline{EA}$ and $\overline{FD}$, and H the intersection of $\overline{EC}$ and $\overline{FD}$. Since $OE = 1$, the height of the equilateral triangle $\triangle EGH$ is $1/2$. It easily follows that $\triangle EGH$ has side-length $\sqrt{3}/3$ and area $\sqrt{3}/12$. The star-shaped region can be divided up into 12 equilateral triangles, each congruent to $\triangle EGH$. It follows that the area of the shaded region is $\sqrt{3}$.

22. **(c)** If g is the number of girls and b is the number of boys, then $g + b = 12$. A team can be chosen by selecting one of the g girls, one of the b boys, and one of the remaining $g + b - 2 = 10$ students. If we consider every such formulation of a team, we will count each team twice (for example, if the first person selected is Cathy and the second person selected is Bob and the third person selected is Dave, this will result in the same team as selecting first Cathy, then Dave, and then Bob). Hence, the total number of teams possible is $5gb$. Since this total is 160, we obtain that $gb = 32$. Given $g + b = 12$ and $gb = 32$, we get that g and b are 8 and 4 in some order. This leads to the answer indicated.

23. **(b)** Observe that by the Pythagorean Theorem, $w^2 + v^2 = x^2$. Also, $vw = xy$ as each of these expressions is equal to twice the area of $\triangle ABC$. Hence,

$$35^2 = (v+w)^2 = u^2 + 2vw + w^2 = x^2 + 2xy = (x+y)^2 - y^2 = 37^2 - y^2.$$

Thus, $y^2 = 37^2 - 35^2 = (37 - 35)(37 + 35) = 2 \cdot 72 = 144$. It follows that $y = 12$.

24. **(c)** Note that $\theta \neq 0^\circ$ and $\theta \neq 90^\circ$. From the given information, $4 \sin\theta \cos\theta \cos(2\theta) = \sin\theta$. Using the identity $\sin(2x) = 2\sin x \cos x$ twice (with $x = \theta$ and with $x = 2\theta$), we deduce $\sin(4\theta) = \sin\theta$. Since $0^\circ < \theta < 90^\circ$, we see that $\sin\theta$ and, hence, $\sin(4\theta)$ are positive. We deduce that $90^\circ < 4\theta < 180^\circ$ and, furthermore, that $\theta = 180^\circ - 4\theta$. This implies $\theta = 180^\circ/5 = 36^\circ$. (One can verify that $\theta = 36^\circ$ is a solution by reversing the steps of the argument.)

25. **(d)** One checks that the first number the two progressions have in common is 57. One can write the elements of the first progression as $7x + 57$ where x is an integer satisfying $-8 \leq x \leq 278$. The elements of the second progression can be written in the form $11y + 57$ where y is an integer satisfying $-5 \leq y \leq 177$. A number n is in both progressions precisely when $n = 7x + 57$ and $n = 11y + 57$ with x and y as indicated. But then $7x + 57 = 11y + 57$ so that $7x = 11y$. This occurs precisely when $x = 11k$ and $y = 7k$ for some integer k. The conditions $-8 \leq 11k \leq 278$ and $-5 \leq 7k \leq 177$ both hold precisely when $0 \leq k \leq 25$. Thus, there are 26 elements in both progressions corresponding to these 26 different values of k.

26. **(a)** There are $\binom{8}{3}5^5$ different outcomes that are possible from the 8 rolls of the die given that 3 occurs exactly three times (obtained from first picking which 3 of the 8 rolls end up with 3 face-up and then choosing one of the 5 remaining numbers for each of the remaining rolls). Next, we count how many of these outcomes do not have two 3's next to each other. This can be done as follows. Imagine 6 apples in a row. Decide on 3 at random for eating purposes, but don't consume them yet. Instead add two more apples to the row, one to the right of each of the two left-most apples you have chosen at random to eat. Now, look at what you have in front of you: 8 apples with 3 in your mind for consumption and no two of these three are next to each other. Thus, each selection of 3 apples from a row of 6 corresponds to a selection of 3 apples from a row of 8 with no two of the 3 chosen apples next to each other. This works backwards too. If you start with 8 apples in a row and pick 3 at random but with the added condition that no two of the 3 chosen are next to each other, then you can remove the apple immediately to the right of the two left-most apples chosen and you will have 3 apples chosen from a row of 6 apples. In mathematical terms, we have a one-to-one correspondence between choosing 3 apples from a row of 6 apples and choosing 3 apples from a row of 8 apples with the added condition in this latter case of not choosing two apples that are next to each other. Now, this problem is not talking about apples or even oranges, but this discussion implies that there are $\binom{6}{3}$ ways of picking 3 out of 8 rolls to end face-up with a 3 given the added condition that no two consecutive 3's are rolled. In addition, there are 5^5 possibilities for the remaining rolls. Recall that we started with $\binom{8}{3}5^5$ different outcomes. We deduce that the number of these that do not involve 3 appearing face-up on two consecutive rolls is $\binom{6}{3}5^5$. Hence, the probability is

$$\frac{\binom{6}{3}5^5}{\binom{8}{3}5^5} = \frac{\binom{6}{3}}{\binom{8}{3}} = \frac{6 \cdot 5 \cdot 4}{8 \cdot 7 \cdot 6} = \frac{5}{14}.$$

Note that the die doesn't need to be standard, doesn't need to be fair, and doesn't need to be 6-sided. The die doesn't really need to be a die. Now, go eat your apples.

27. **(d)** For each $x \in \{101, 102, \ldots, 299\}$, there is at most one integer y satisfying $(x/3)+0.1 < y < (x/3)+0.6$ (since the difference between the upper and lower bounds is 0.5). Furthermore, if x is of the form $3k$ for some integer k, then there are no such integers y since the double inequality becomes $k+0.1 < y < k+0.6$. Similarly, if $x = 3k+1$ for some integer k, then there are no such y as the double inequality becomes $k+(1/3)+0.1 < y < k+(1/3)+0.6$. On the other hand, if $x = 3k+2$ for some integer k, then the double inequality becomes $k+(2/3)+0.1 < y < k+(2/3)+0.6$ and there is exactly one such integer y. Thus, the answer is the number of x of the form $3k+2$ in $\{101, 102, \ldots, 299\}$. These x correspond to $k \in \{33, 34, 35, \ldots, 99\}$. Hence, there are 67 such integral (x, y).

28. **(d)** We use $\mathcal{A}(\triangle UVW)$ to denote the area of a triangle with vertices at U, V and W. Let $r = \mathcal{A}(\triangle DAO)$, and let $s = \mathcal{A}(\triangle BOC)$. Then a well-known identity gives that $rs = 4 \cdot 9 = 36$. This can be shown as follows. Let h_D denote the length of the altitude of $\triangle DAO$ drawn from D, and let h_B denote the length of the altitude of $\triangle BOC$ drawn from B. Then

$$\mathcal{A}(\triangle DAO) \cdot \mathcal{A}(\triangle BOC) = \frac{h_D \cdot AO}{2} \cdot \frac{h_B \cdot OC}{2} = \frac{h_D \cdot OC}{2} \cdot \frac{h_B \cdot AO}{2} = \mathcal{A}(\triangle DOC) \cdot \mathcal{A}(\triangle BAO).$$

The result $rs = 4 \cdot 9 = 36$ follows. We deduce that the sum of the areas of $\triangle DAO$ and $\triangle BOC$ is $r + 36/r$. The arithmetic-geometric mean inequality asserts that $(a+b)/2 \geq \sqrt{ab}$ for any positive numbers a and b. Taking $a = r$ and $b = 36/r$, we see that $r + 36/r \geq 2\sqrt{36} = 12$. Thus, the sum of the areas of $\triangle DAO$ and $\triangle COB$ is at least 12. It follows that the area of quadrilateral $ABCD$ is at least $4 + 9 + 12 = 25$.

To see that it is possible for quadrilateral $ABCD$ to have area 25, consider the situation where $\triangle COD$ is an isosceles right triangle with hypotenuse DC and legs each of length $2\sqrt{2}$ and where $\triangle AOB$ is an isosceles right triangle with hypotenuse AB and legs each of length $3\sqrt{2}$. Then the area of $\triangle COD$ is 4 and the area of $\triangle AOB$ is 9. Also, the areas of $\triangle DAO$ and $\triangle BOC$ are each 6. Thus, in this case, the area of quadrilateral $ABCD$ is indeed 25.

29. **(b)** Observe that

$$a = 3 \cdot \frac{10^{2003} - 1}{9} \quad \text{and} \quad b = 6 \cdot \frac{10^{2003} - 1}{9}.$$

It follows that

$$ab = \frac{2(10^{2003} - 1)^2}{9} = \frac{2 \cdot 10^{2003} \cdot (10^{2003} - 1)}{9} - \frac{2 \cdot (10^{2003} - 1)}{9}.$$

The decimal expansion of the first fraction on the right is 2003 consecutive 2's followed by 2003 consecutive 0's. The decimal expansion of the second fraction on the right is simply 2003 consecutive 2's. It follows that the decimal expansion of ab is 2002 consecutive 2's (on the left), followed by a 1, followed by 2002 consecutive 7's, and finally followed by an 8. The 2004^{th} digit from the right is a 1.

30. **(c)** Let $n+1, n+2, \ldots, n+k$ be k consecutive integers that sum to 2004. Since the sum of these numbers is also $kn + k(k+1)/2$, we deduce

$$kn + \frac{k(k+1)}{2} = 2004 \qquad (*)$$

is a necessary and sufficient condition for the sum of the numbers $n+1, n+2, \ldots, n+k$ to be 2004. We make some observations based on $(*)$. Since $k(k+1)/2 > 2004$ for $k \geq 63$, we see that $k < 63$. If $k = 2\ell$ with ℓ odd, then kn and 2004 are even and $k(k+1)/2$ is odd which implies $(*)$ cannot hold. If $k = 4\ell$ with ℓ odd, then kn and 2004 are both divisible by 4 and $k(k+1)/2$ is not which implies $(*)$ cannot hold. Thus, either k is odd or 8 divides k. Now, if k is odd, then k divides the left-hand side of $(*)$ and, hence, 2004. If k is even, then we get similarly that $k/2$ divides 2004. As $2004 = 2^2 \cdot 3 \cdot 167$ is the prime decomposition of 2004, we see that either k is an odd divisor of 12 or k is a multiple of 8 that divides 24. By the conditions in the problem, $k > 1$. We deduce that k must be one of 3, 8, and 24. We are given an example with $k = 3$. Examples with $k = 8$ and $k = 24$ also exist and can be found directly from $(*)$. For $k = 8$, we get from $(*)$ that $n = 246$ (so that the sum of the numbers from 247 to 254, inclusive, is 2004). For $k = 24$, we get from $(*)$ that $n = 71$ (so that the sum of the numbers from 72 to 95, inclusive, is 2004). Thus, the answer is 3.