

# Multigrid Methods for Fourth Order Problems

by

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# Abstract

In the dissertation we discuss multigrid methods for fourth order partial differential equations. We choose the biharmonic problem as the model problem. Multigrid methods using finite element discretizations such as the Morley element, the Hsieh-Clough-Tocher (HCT) element, the reduced HCT element and the incomplete biquadratic element, are discussed. Using the additive theory for the convergence of V-cycle and F-cycle multigrid methods, we prove that the methods are convergent, and the contraction numbers are uniformly decreasing as the number of smoothing steps increases. We show that, for  $m$  sufficiently large, the the contraction numbers are less than or equal to  $C/m^{\alpha/2}$ , where  $m$  is the number of pre-smoothing and post-smoothing steps,  $\alpha$  is the index of elliptic regularity, and  $C$  is a mesh-independent constant.

Numerical results are also presented to support the theory. Performances of different algorithms are compared and some suggestions to improve the performances of the algorithms are given.

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# Chapter 1

## Introduction

### 1.1 The Biharmonic Problem

Let  $\Omega \subset \mathbb{R}^2$  be a bounded polygonal domain. We consider the following biharmonic problem with homogeneous Dirichlet boundary conditions:

Find  $u \in H_0^2(\Omega)$  such that

$$\begin{aligned}\Delta^2 u &= f && \text{in } \Omega, \\ u &= \partial u / \partial n = 0 && \text{on } \partial\Omega,\end{aligned}\tag{1.1.1}$$

where  $f \in L_2(\Omega)$  and we use the notations of Sobolev spaces in [1].

The biharmonic problem models plate bending. We will consider it as our model problem in this dissertation.

By multiplying a test function  $v \in H_0^2(\Omega)$  on both sides of (1.1.1) and using integration by parts, we have the following variational form of the biharmonic problem:

Find  $u \in H_0^2(\Omega)$  such that

$$a(u, v) = \phi(v) \quad \forall v \in H_0^2(\Omega),\tag{1.1.2}$$

where

$$a(u, v) = \int_{\Omega} D^2 u : D^2 v \, dx := \int_{\Omega} \sum_{i,j=1}^2 \frac{\partial^2 u}{\partial x_i \partial x_j} \cdot \frac{\partial^2 v}{\partial x_i \partial x_j} \, dx,$$

and

$$\phi(v) = \int_{\Omega} f v \, dx.$$

The Cauchy-Schwarz inequality implies that the bilinear form  $a(\cdot, \cdot)$  is bounded, i.e.,

$$|a(v, w)| \leq \|v\|_{H^2(\Omega)} \|w\|_{H^2(\Omega)} \quad \forall v, w \in H_0^2(\Omega). \quad (1.1.3)$$

The Poincaré inequality implies that the bilinear form is coercive, i.e.,

$$|a(v, v)| \geq C \|v\|_{H^2(\Omega)}^2 \quad \forall v \in H_0^2(\Omega) \quad (1.1.4)$$

for some constant  $C$ . Therefore by the Riesz Representation Theorem we know that there exists a unique  $u \in H_0^2(\Omega)$  that solves (1.1.2).

Moreover  $\phi$  can be chosen from a more general space  $H^{-2}(\Omega) = [H_0^2(\Omega)]'$ . Elliptic regularity of the biharmonic equation (cf. [3, 33]) implies that there exists  $\alpha \in (\frac{1}{2}, 1]$  such that the solution  $u$  of (1.1.2) belongs to  $H^{2+\alpha}(\Omega) \cap H_0^2(\Omega)$  whenever  $\phi \in H^{-2+\alpha}(\Omega)$  and

$$\|u\|_{H^{2+\alpha}(\Omega)} \leq C_{\Omega} \|\phi\|_{H^{-2+\alpha}(\Omega)}, \quad (1.1.5)$$

where  $C_{\Omega}$  depends only on the shape of  $\Omega$ .

Finally the following duality estimate holds for the bilinear form  $a(\cdot, \cdot)$  (cf. [24]):

$$a(v, w) \leq C |v|_{H^{2+\alpha}(\Omega)} |w|_{H^{2-\alpha}(\Omega)} \quad \forall v \in H^{2+\alpha}(\Omega) \cap H_0^2(\Omega), w \in H_0^2(\Omega), \quad (1.1.6)$$

for some constant  $C$ .

## 1.2 Finite Element Spaces and Multigrid Methods

Numerical solutions of the model problem can be obtained using finite elements methods based on the Bogner-Fox-Schmit element (cf. [8]), the Argyris element (cf. [2]),

the Hsieh-Clough-Tocher element (cf. [32]), the Morley element (cf. [42]) and the incomplete biquadratic element (cf. [45]).

Let  $\{\mathcal{T}_k\}_{k \geq 1}$  be a family of triangulations of  $\Omega$  obtained by regular subdivision, i.e.,  $\mathcal{T}_{k+1}$  is obtained by connecting the midpoints of edges of the triangles (or rectangles) in  $\mathcal{T}_k$ . We denote the mesh size of  $\mathcal{T}_k$  by  $h_k = \max\{\text{diam } T : T \in \mathcal{T}_k\}$ . Note that

$$h_{k-1} = 2h_k. \quad (1.2.1)$$

Let  $V_k$  be a finite element space associated the triangulation  $\mathcal{T}_k$ . If  $V_k \subset H_0^2(\Omega)$ , we say the finite element is conforming. If  $V_k \not\subset H_0^2(\Omega)$ , the finite element is nonconforming.

Let  $a_k(\cdot, \cdot)$  be the extension of  $a(\cdot, \cdot)$  to the space  $H_0^2(\Omega) + V_k$  defined by

$$a_k(v, w) := \sum_{T \in \mathcal{T}_k} \int_T D^2 v : D^2 w \, dx, \quad \forall v, w \in H_0^2(\Omega) + V_k. \quad (1.2.2)$$

Then an approximate solution of (1.1.2) can be obtained by the following finite element method:

Find  $u_k \in V_k$  such that

$$a_k(u_k, v) = \int_{\Omega} f v \, dx \quad \forall v \in V_k. \quad (1.2.3)$$

Let  $(\cdot, \cdot)_k$  be a discrete inner product on  $V_k$ . Then we can define  $A_k : V_k \longrightarrow V_k$  by

$$(A_k v, w)_k = a_k(v, w) \quad \forall v, w \in V_k, \quad (1.2.4)$$

and rewrite (1.2.3) as

$$A_k u_k = f_k, \quad (1.2.5)$$

where  $f_k \in V_k$  is defined by

$$(f_k, v)_k = \int_{\Omega} f v \, dx \quad \forall v \in V_k.$$

Using a basis of  $V_k$ , the problem (1.2.5) is represented as a system of linear equations.  $A_k$  is a sparse matrix with a very large condition number. Classical iterative methods converge very slowly for such systems. Multigrid methods, which are fast solvers for (1.2.5), can overcome this difficulty. In this dissertation we choose the following finite elements to illustrate the multigrid theory.

- *The Hsieh-Clough-Tocher (HCT) element and the reduced HCT element*

The HCT and the reduced HCT macro elements are conforming methods. The approximate solutions obtained by these methods belong to  $H_0^2(\Omega)$ .

- *The Morley finite element*

The Morley element is nonconforming, but is the simplest among all triangular finite elements for fourth order problems.

- *Incomplete biquadratic element*

This nonconforming element is the simplest among all rectangular elements suitable for multigrid solvers.

We will solve (1.2.5) by the following multigrid methods.

**The  $k$ th level multigrid iteration.** The  $k$ th level multigrid iteration yields  $MG(k, g, z_0, m_1, m_2)$  as an approximate solution to the equation

$$A_k z = g, \tag{1.2.6}$$

where  $z_0$  is an initial guess and  $m_1$  and  $m_2$  are the numbers of pre-smoothing and post-smoothing steps respectively.

For  $k = 1$ ,  $MG(k, g, z_0, m_1, m_2) = A_1^{-1}g$ .

For  $k > 1$ ,  $MG(k, g, z_0, m_1, m_2)$  is obtained by the following three steps:

1. *Pre-smoothing*

For  $j = 1, 2, \dots, m_1$ , compute  $z_j$  by

$$z_j = z_{j-1} + \frac{1}{\Lambda_k}(g - A_k z_{j-1}).$$

2. *Coarse Grid Correction*

Let  $r_k = g - A_k z_{m_1}$  be the residual. Apply the  $(k - 1)$ st level iteration  $p$  times with initial guess 0 to the equation

$$A_{k-1} r_{k-1} = I_k^{k-1} r_k$$

to obtain  $\tilde{r}_{k-1} \in V_{k-1}$  and make the error correction

$$z_{m_1+1} = z_{m_1} + I_{k-1}^k \tilde{r}_{k-1}.$$

3. *Post-smoothing*

For  $j = m_1 + 2, m_1 + 3, \dots, m_1 + m_1 + 1$ , compute  $z_j$  by

$$z_j = z_{j-1} + \frac{1}{\Lambda_k}(g - A_k z_{j-1}).$$

Finally we set  $MG(k, g, z_0, m_1, m_2) = z_{m_1+m_2+1}$ .

In the algorithm above, we use Richardson relaxation as the smoother for simplicity. Other smoothers may also apply (cf. [6, 14, 15, 25]). If we denote the spectral radius of  $A_k$  as  $\rho(A_k)$ , then  $\Lambda_k$  is chosen so that

$$\rho(A_k) < \Lambda_k = Ch_k^{-4} \tag{1.2.7}$$

for some mesh-independent constant  $C$ . The operators  $I_k^{k-1} : V_k \longrightarrow V_{k-1}$  and  $I_{k-1}^k : V_{k-1} \longrightarrow V_k$  are called fine-to-coarse and coarse-to-fine intergrid transfer operators respectively.

If  $p = 1$ , the algorithm is called a V-cycle. If  $p = 2$  we call it a W-cycle algorithm. If, in the coarse grid error correction step, we use the  $(k - 1)$ st level iteration once,

followed by a V-cycle iteration, the algorithm is called an F-cycle. The V-cycle, F-cycle and W-cycle algorithms are denoted by  $MG_{\mathcal{V}}$ ,  $MG_{\mathcal{F}}$  and  $MG_{\mathcal{W}}$  respectively in this dissertation. For a V-cycle algorithm, if the numbers of smoothing steps at different levels are different, and the number  $m(k)$  of smoothing steps at the  $k$ th level satisfies

$$\beta_0 m(k) \leq m(k-1) \leq \beta_1 m(k)$$

with  $1 < \beta_0 \leq \beta_1$ , the algorithm is called a variable V-cycle algorithm.

### 1.3 Historical Background and the Results in This Dissertation

Multigrid methods were first studied in Russia in the sixties (cf. [4, 34]) and were made popular in the west by Brandt in the seventies (cf. [15, 19, 36, 41, 47, 51]).

The first convergence result for W-cycle algorithms for multigrid methods for conforming finite element discretizations was obtained by Bank and Dupont in 1981 (cf. [5]).

In 1983 Braess and Hackbusch proved in [9], under the assumption of full elliptic regularity, that the contraction numbers (in the energy norm) of the symmetric V-cycle multigrid algorithm applied to second order elliptic boundary value problems are less than or equal to  $C/(C+m)$ , where  $m$  is the number of pre-smoothing (and post-smoothing) steps and the constant  $C$  is independent of the number of grid levels.

The results of Bank-Dupont and Braess-Hackbusch were the most important results in multigrid theory in the eighties.

The case of less than full elliptic regularity is much more subtle. It was not until 1992, after a multiplicative multigrid theory had been developed, that Zhang, Bramble and Pasciak, Xu and others (cf. [11, 13, 15, 16, 17, 18, 53]) showed that the contraction numbers are less than or equal to a number  $\delta$  between 0 and 1,

independent of the number of smoothing steps and grid levels. This result stood as the best achievement in multigrid theory throughout the nineties. However, it is not a complete generalization of the result of Braess and Hackbusch because it does not show that the contraction numbers will decrease if  $m$  is increased. This important difference was pointed out by Bramble in his famous book on multigrid (cf. [15]).

In [26] Brenner developed an additive multigrid theory and showed that the contraction numbers are less than or equal to  $C/(C+m^\alpha)$ , where  $\alpha$  is the index of elliptic regularity ( $\alpha = 1$  in the case of full elliptic regularity), and thus established the long sought-after complete generalization of the classical result of Braess and Hackbusch.

Nonconforming finite elements provide some of the simplest numerical methods for complicated problems. But their analyses require more sophisticated techniques. The first convergence analysis of W-cycle multigrid methods for nonconforming finite elements for second and fourth order problems (under the assumption of full elliptic regularity) can be found in Brenner's Ph.D. dissertation [20]. These results were generalized to the case of less than full elliptic regularity in [24], where variable V-cycle algorithms were also shown to be optimal preconditioners.

The additive multigrid theory was generalized by Brenner to nonconforming finite element methods in [28], where it was shown that, for V-cycle and F-cycle algorithms for nonconforming finite element methods, the contraction numbers are less than or equal to  $C/m^\alpha$ , provided  $m$  is sufficiently large. They were the first general results for such algorithms. F-cycle algorithms using nonconforming finite elements had been employed successfully in large scale computations for fluid flow problems (cf. [49]), but no convergence analysis had ever been carried out until [28].

For fourth order problems, the simplest finite element methods are nonconforming or conforming but nonnested (which means the finite element spaces on the coarser grids are not subspaces of the ones on finer grids). Multigrid methods for fourth order

problems were studied in [17, 21, 24, 37, 43, 46, 50, 52]. In these works, only W-cycle and variable V-cycle algorithms were discussed, or conforming and nested finite elements were involved. Until recently, there were no results on V-cycle and F-cycle algorithms based on nonconforming or conforming but nonnested finite elements.

In this dissertation, the additive multigrid theory is extended to fourth order problems. We will describe the theory in the next chapter. In the following chapters we will apply the theory to the HCT and the reduced HCT macro elements (cf. [55]), the Morley finite element (cf. [54]) and the incomplete biquadratic element.

Specifically, the following tasks will be performed for each finite element:

- *Design of the algorithms*

To apply the multigrid algorithms described in the previous section, we need to define the discrete inner products on the finite element spaces and the intergrid transfer operators.

- *Convergence analysis*

The convergence of V-cycle and F-cycle algorithms for sufficiently large number of smoothing steps will be established, and the contraction numbers of these algorithms are to be shown to decrease at a rate determined by the index of elliptic regularity.

- *Numerical results*

Numerical results will also be presented to illustrate the theoretical conclusions. Performances of different algorithms will be compared and some suggestions to improve the performances of the algorithms will be given.

# Chapter 2

## The Additive Theory

### 2.1 Assumptions of the Additive Theory

Let  $\mathbb{E}_{k,m} : V_k \longrightarrow V_k$  be the error propagation operator of the symmetric V-cycle algorithm applied to the equation (1.2.6), i.e.,

$$\mathbb{E}_{k,m}(z - z_0) = z - MG_V(k, g, z_0, m, m), \quad (2.1.1)$$

where  $z$  is the exact solution of (1.2.6). The following relations (cf. [15] and [36]) are well-known:

$$\mathbb{E}_{k,m} = R_k^m [(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k \mathbb{E}_{k-1,m} P_k^{k-1}] R_k^m \quad \text{for } k \geq 2, \quad (2.1.2)$$

$$\mathbb{E}_{1,m} = 0. \quad (2.1.3)$$

From (2.1.2) and (2.1.3) we can also obtain an additive expression for  $\mathbb{E}_{k,m}$  (cf. [26]):

$$\begin{aligned} \mathbb{E}_{k,m} &= R_k^m [(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k \mathbb{E}_{k-1,m} P_k^{k-1}] R_k^m \\ &= R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m \\ &\quad + R_k^m I_{k-1}^k R_{k-1}^m [(Id_{k-1} - I_{k-2}^{k-1} P_{k-1}^{k-2}) \\ &\quad\quad\quad + I_{k-2}^{k-1} \mathbb{E}_{k-2,m} P_{k-1}^{k-2}] R_{k-1}^m P_k^{k-1} R_k^m \\ &= \sum_{j=2}^k T_{k,j,m} R_j^m (Id_j - I_{j-1}^j P_j^{j-1}) R_j^m T_{j,k,m}, \end{aligned} \quad (2.1.4)$$

where  $Id_k : V_k \longrightarrow V_k$  is the identity operator on  $V_k$ , the operator  $R_k : V_k \longrightarrow V_k$  is defined by

$$R_k = Id_k - \frac{1}{\Lambda_k} A_k, \quad (2.1.5)$$

$P_k^{k-1} : V_k \longrightarrow V_{k-1}$  is the transpose of  $I_{k-1}^k$  with respect to the bilinear form  $a_k(\cdot, \cdot)$ , i.e.,

$$a_{k-1}(P_k^{k-1}v, w) = a_k(v, I_{k-1}^k w) \quad \forall v \in V_k, w \in V_{k-1}. \quad (2.1.6)$$

The operators  $T_{j,k,m} : V_k \longrightarrow V_j$  and  $T_{k,j,m} : V_j \longrightarrow V_k$  are defined by

$$T_{j,k,m} = P_{j+1}^j R_{j+1}^m \cdots P_k^{k-1} R_k^m, \quad (2.1.7)$$

$$T_{k,j,m} = R_m^m I_{k-1}^k \cdots R_{j+1}^m I_j^{j+1}, \quad (2.1.8)$$

for  $j < k$ , and

$$T_{k,k,m} = Id_k. \quad (2.1.9)$$

It is easy to see that

$$a_j(R_j v, w) = a_j(v, R_j w) \quad \forall v, w \in V_j, \quad (2.1.10)$$

$$a_j(T_{j,k,m} v, w) = a_k(v, T_{k,j,m} w) \quad \forall v \in V_k, w \in V_j, \quad (2.1.11)$$

and for  $1 \leq j \leq k \leq l$  the following relations are valid:

$$T_{j,l,m} = T_{j,k,m} T_{k,l,m} \text{ and } T_{l,j,m} = T_{l,k,m} T_{k,j,m}. \quad (2.1.12)$$

Let  $\tilde{\mathbb{E}}_{k,m} : \tilde{V}_k \longrightarrow \tilde{V}_k$  be the error propagation operator of the symmetric F-cycle algorithm applied to the equation (1.2.6), i.e.,

$$\tilde{\mathbb{E}}_{k,m}(z - z_0) = z - MG_{\mathcal{F}}(k, g, z_0, m), \quad (2.1.13)$$

where  $z$  is the exact solution of (1.2.6). The following relations are also well-known (cf. [47]):

$$\tilde{\mathbb{E}}_{1,m} = 0 \quad (2.1.14)$$

$$\tilde{\mathbb{E}}_{k,m} = R_k^m [(Id_k - I_{k-1}^k P_k^{k-1}) + I_{k-1}^k \tilde{\mathbb{E}}_{k-1,m} \tilde{\mathbb{E}}_{k-1,m} P_k^{k-1}] R_k^m, \quad k \geq 2. \quad (2.1.15)$$

An additive theory for the convergence analysis of V-cycle and F-cycle multigrid algorithms is developed in [28] based on the expressions (2.1.4) and (2.1.15).

An important tool for the convergence analysis is the mesh-dependent norm  $\|\cdot\|_{s,k}$  defined as follows:

$$\|v\|_{s,k} = \sqrt{(A_k^{s/2}v, v)_k} \quad \forall v \in V_k. \quad (2.1.16)$$

According to the theory, we need to verify the following assumptions to complete the convergence analysis.

*Assumptions on  $V_k$ :*

$$(v, v)_k \approx \|v\|_{L_2(\Omega)}^2 \quad \forall v \in V_k, \quad (2.1.17)$$

$$\|v\|_{a_k} \lesssim h_k^{-2} \|v\|_{L_2(\Omega)} \quad \forall v \in V_k, \quad (2.1.18)$$

where the energy norm  $\|\cdot\|_{a_k}$  in the assumption (2.1.18) is defined by

$$\|v\|_{a_k} = \sqrt{a_k(v, v)} \quad \forall v \in V_k. \quad (2.1.19)$$

Note from (1.2.2) that  $a_k(v, v)$  is well defined for  $v \in V_{k-1}$  and

$$a_k(v, v) = a_{k-1}(v, v) \quad \forall v \in V_{k-1}. \quad (2.1.20)$$

To avoid the proliferation, we use  $X \lesssim Y$  for two expressions  $X$  and  $Y$  to denote that  $X \leq CY$  for some constant  $C$  that is mesh-independent, and  $X \approx Y$  means  $X \lesssim Y$  as well as  $Y \lesssim X$ .

*Assumptions on  $I_{k-1}^k$  and  $P_k^{k-1}$ :*

$$\begin{aligned} \|I_{k-1}^k v\|_{2,k}^2 &\leq (1 + \theta^2) \|v\|_{2,k-1}^2 + C_1 \theta^{-2} h_k^{2\alpha} \|v\|_{2+\alpha,k-1}^2 \\ &\quad \forall v \in V_{k-1}, \theta \in (0, 1), \end{aligned} \quad (2.1.21)$$

$$\begin{aligned} \|I_{k-1}^k v\|_{2-\alpha,k}^2 &\leq (1 + \theta^2) \|v\|_{2-\alpha,k-1}^2 + C_2 \theta^{-2} h_k^{2\alpha} \|v\|_{2,k-1}^2 \\ &\quad \forall v \in V_{k-1}, \theta \in (0, 1), \end{aligned} \quad (2.1.22)$$

$$\begin{aligned} \|P_k^{k-1} v\|_{2-\alpha,k-1}^2 &\leq (1 + \theta^2) \|v\|_{2-\alpha,k}^2 + C_3 \theta^{-2} h_k^{2\alpha} \|v\|_{2,k}^2 \\ &\quad \forall v \in V_k, \theta \in (0, 1), \end{aligned} \quad (2.1.23)$$

where the constants  $C_1, C_2$  and  $C_3$  are mesh-independent.

*Assumptions on  $I_{k-1}^k P_k^{k-1}$  and  $P_k^{k-1} I_{k-1}^k$ :*

$$\| (Id_k - I_{k-1}^k P_k^{k-1}) v \|_{2-\alpha, k} \lesssim h_k^{2\alpha} \| v \|_{2+\alpha, k} \quad \forall v \in V_k, \quad (2.1.24)$$

$$\| (Id_{k-1} - P_k^{k-1} I_{k-1}^k) v \|_{2-\alpha, k-1} \lesssim h_k^\alpha \| v \|_{2, k-1} \quad \forall v \in V_{k-1}. \quad (2.1.25)$$

Based on the these assumptions, we will establish the convergence of V-cycle and F-cycle algorithms.

### 2.1.1 Preliminary Estimates

First we consider the operator  $A_k : V_k \longrightarrow V_k$  defined by (1.2.4). It is easy to see that  $A_k$  is symmetric positive definite with respect to the discrete inner product  $(\cdot, \cdot)_k$  and the following relation holds:

$$a_k(A_k^s v_1, v_2) = a_k(v_1, A_k^s v_2) \quad \forall v_1, v_2 \in V_k, s \in \mathbb{R}. \quad (2.1.26)$$

Moreover, let  $\lambda$  be an arbitrary eigenvalue of  $A_k$  and  $v \in V_k$  be a corresponding eigenvector. Then

$$a_k(v, v) = (A_k v, v)_k = \lambda (v, v)_k.$$

From assumptions (2.1.17) and (2.1.18) we have

$$\lambda = \frac{a_k(v, v)}{(v, v)_k} \lesssim h_k^{-4}.$$

Therefore the spectral radius  $\rho(A_k)$  of  $A_k$  satisfies

$$\rho(A_k) \lesssim h_k^{-4}. \quad (2.1.27)$$

Now we discuss the properties of the mesh-dependent norms. It is easy to see from (1.2.4), (2.1.16) and (2.1.17) that

$$\| v \|_{0, k} = \sqrt{(v, v)_k} \approx \| v \|_{L_2(\Omega)} \quad \forall v \in V_k, \quad (2.1.28)$$

$$\| v \|_{2, k} = \sqrt{a_k(v, v)} = \| v \|_{a_k} \quad \forall v \in V_k, \quad (2.1.29)$$

$$\| A_k^s v \|_{t, k} = \| v \|_{t+4s, k} \quad \forall v \in V_k, s, t \in \mathbb{R}. \quad (2.1.30)$$

Moreover the Cauchy-Schwarz inequality implies that

$$\|v\|_{2+t,k} = \sup_{w \in V_k \setminus \{0\}} \frac{a_k(v, w)}{\|w\|_{2-t,k}} \quad \forall t \in \mathbb{R}, v \in V_k. \quad (2.1.31)$$

Let  $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{n_k}$  be the eigenvalues of the operator  $A_k$  and  $\phi_1, \phi_2, \dots, \phi_{n_k}$  be the corresponding eigenvectors satisfying the orthonormal relation  $(\phi_i, \phi_j)_k = \delta_{ij}$ , where  $\delta_{ij}$  is the Kronecker delta, i.e.,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

For any  $v \in V_k$ , we can write  $v = \sum_{i=1}^{n_k} c_i \phi_i$ . Then  $(v, v)_k = \sum_{i=1}^{n_k} c_i^2$  and from (2.1.27) we have

$$\begin{aligned} (A_k^s v, v)_k &= \left( \sum_{i=1}^{n_k} \lambda_i^s c_i \phi_i, \sum_{i=1}^{n_k} c_i \phi_i \right)_k = \sum_{i=1}^{n_k} \lambda_i^s c_i^2 \\ &\lesssim h_k^{-4s} \sum_{i=1}^{n_k} c_i^2 = h_k^{-4s} (v, v)_k \end{aligned}$$

for all  $s \geq 0$ . Therefore we have

$$\begin{aligned} \|v\|_{s,k}^2 &= (A_k^{s/2} v, v)_k \\ &= (A_k^{s/2-t/2} A_k^{t/4} v, A_k^{t/4} v)_k \\ &\lesssim h_k^{-2(s-t)} (A_k^{t/4} v, A_k^{t/4} v)_k \\ &= h_k^{-2(s-t)} (A_k^{t/2} v, v)_k = h_k^{-2(s-t)} \|v\|_{t,k}^2 \end{aligned}$$

for  $s \geq t$ . Thus the following inverse estimate is established:

$$\|v\|_{s,k} \lesssim h_k^{-(s-t)} \|v\|_{t,k} \quad \forall v \in V_k, 0 \leq t \leq s. \quad (2.1.32)$$

Let  $R_k : V_k \longrightarrow V_k$  be defined as (2.1.5). We have the following lemma.

**Lemma 2.1** *The following smoothing properties hold for  $0 \leq s \leq t \leq 4$ :*

$$\|R_k v\|_{s,k} \leq \|v\|_{s,k} \quad \forall v \in V_k, \quad (2.1.33)$$

$$\|R_k^m v\|_{s,k} \lesssim h_k^{-(s-t)} m^{-(s-t)/4} \|v\|_{t,k} \quad \forall v \in V_k. \quad (2.1.34)$$

*Proof.* Let  $v \in V_k$  and  $v = \sum_{i=1}^{n_k} c_i \phi_i$ . Then

$$R_k v = \sum_{i=1}^{n_k} \left(1 - \frac{\lambda_i}{\Lambda_k}\right) c_i \phi_i.$$

Therefore

$$\begin{aligned} \|R_k v\|_{s,k}^2 &= \left( A_k^{s/2} \sum_{i=1}^{n_k} \left(1 - \frac{\lambda_i}{\Lambda_k}\right) c_i \phi_i, \sum_{i=1}^{n_k} \left(1 - \frac{\lambda_i}{\Lambda_k}\right) c_i \phi_i \right)_k \\ &= \sum_{i=1}^{n_k} \lambda_i^{s/2} \left(1 - \frac{\lambda_i}{\Lambda_k}\right)^2 c_i^2 \\ &\leq \sum_{i=1}^{n_k} \lambda_i^{s/2} c_i^2 = \|v\|_{s,k}^2. \end{aligned}$$

The estimate (2.1.33) then follows.

Moreover from (1.2.7) and (2.1.27) we have,

$$\begin{aligned} \|R_k^m v\|_{s,k}^2 &= \sum_{i=1}^{n_k} \lambda_i^{s/2} \left(1 - \frac{\lambda_i}{\Lambda_k}\right)^{2m} c_i^2 \\ &= \Lambda_k^{(s-t)/2} \sum_{i=1}^{n_k} \left(1 - \frac{\lambda_i}{\Lambda_k}\right)^{2m} \left(\frac{\lambda_i}{\Lambda_k}\right)^{(s-t)/2} \lambda_i^{t/2} c_i^2 \\ &\lesssim h_k^{-2(s-t)} \left[ \sup_{0 \leq x \leq 1} (1-x)^{2m} x^{(s-t)/2} \right] \sum_{i=1}^{n_k} \lambda_i^{t/2} c_i^2 \\ &\lesssim h_k^{-2(s-t)} m^{-(s-t)/2} \|v\|_{t,k}^2, \end{aligned}$$

which proves the estimate (2.1.34). □

**Remark 2.2** Corresponding to (2.1.34), the smoothing property for second order problem is slightly different (cf., [28]):

$$\|R_k^m v\|_{s,k} \lesssim h_k^{-(s-t)} m^{-(s-t)/2} \|v\|_{t,k} \quad \forall v \in V_k.$$

Specifically, the exponents for  $m$  are different. We can see that Richardson smoothing for fourth order problems is much less effective than for second order problems.

From the assumptions (2.1.21), (2.1.25), (2.1.31), (2.1.32) and a duality argument (cf. [27]) we have

$$\|I_{k-1}^k v\|_{2,k} \lesssim \|v\|_{2,k-1} \quad \forall v \in V_{k-1}, \quad (2.1.35)$$

$$\|(Id_{k-1} - P_k^{k-1} I_{k-1}^k)v\|_{2,k-1} \lesssim h_k^\alpha \|v\|_{2+\alpha,k-1} \quad \forall v \in V_{k-1}. \quad (2.1.36)$$

From (2.1.6), (2.1.31) and (2.1.35) we have

$$\begin{aligned} a_{k-1}(P_k^{k-1}v, w) &= a_k(v, I_{k-1}^k w) \\ &\lesssim \|v\|_{2,k} \|I_{k-1}^k w\|_{2,k} \\ &\lesssim \|v\|_{2,k} \|w\|_{2,k} \end{aligned}$$

for all  $v \in V_k$  and  $w \in V_{k-1}$ . Applying a duality argument using (2.1.31) we have

$$\|P_k^{k-1}v\|_{2,k-1} \lesssim \|v\|_{2,k} \quad \forall v \in V_k. \quad (2.1.37)$$

**Lemma 2.3** *Given any number  $\omega \in (0, 1)$ , the following estimates holds for sufficiently large  $m$  :*

$$\|I_{k-1}^k R_{k-1}^m v\|_{2,k} \leq (1 + \omega) \|v\|_{2,k-1} \quad \forall v \in V_{k-1}, \quad (2.1.38)$$

$$\|P_k^{k-1} R_k^m v\|_{2-\alpha,k-1} \leq (1 + \omega) \|v\|_{2-\alpha,k} \quad \forall v \in V_k, \quad (2.1.39)$$

$$\|R_k^m I_{k-1}^k v\|_{2+\alpha,k} \leq (1 + \omega) \|v\|_{2+\alpha,k-1} \quad \forall v \in V_{k-1}. \quad (2.1.40)$$

*Proof.* Let  $v \in V_{k-1}$ . From (1.2.1), (2.1.21), (2.1.33) and (2.1.34),

$$\begin{aligned} \|I_{k-1}^k R_{k-1}^m v\|_{2,k} &\leq (1 + \omega^2) \|R_{k-1}^m v\|_{2,k-1}^2 + C\omega^{-2} h_k^{-2\alpha} \|R_{k-1}^m v\|_{2+\alpha,k-1}^2 \\ &\leq (1 + \omega^2) \|v\|_{2,k-1}^2 + C'\omega^{-2} m^{-\alpha/2} \|v\|_{2,k-1}^2, \end{aligned}$$

where the constant  $C'$  is independent of the mesh and the number of smoothing steps.

The estimate (2.1.38) follows if

$$m^{\alpha/2} \geq \frac{C'\omega^{-3}}{2}.$$

The estimates (2.1.39) and (2.1.40) can be similarly obtained.  $\square$

From (2.1.7)–(2.1.9), (2.1.39) and (2.1.40) we have the following corollary.

**Corollary 2.4** *Let  $j \leq k$ . Given any  $\omega \in (0, 1)$  the following estimates hold for sufficiently large  $m$  :*

$$\| \|T_{j,k,m}v\| \|_{2-\alpha,j} \leq (1 + \omega)^{k-j} \| \|v\| \|_{2-\alpha,k} \quad \forall v \in V_k, \quad (2.1.41)$$

$$\| \|T_{k,j,m}v\| \|_{2+\alpha,k} \leq (1 + \omega)^{k-j} \| \|v\| \|_{2+\alpha,j} \quad \forall v \in V_j. \quad (2.1.42)$$

## 2.2 A Strengthened Cauchy-Schwarz Inequality

In this section we will derive some new estimates for  $T_{k,j,m}$  and  $T_{j,k,m}$ . For simplicity we will sometimes suppress the parameter  $m$  and write  $T_{k,j}$  and  $T_{j,k}$ . The constant  $C$  in the proof is mesh-independent and the values of  $C$  at different appearances are not necessarily identical. We first recall two simple inequalities.

$$2ab \leq \theta^2 a^2 + \theta^{-2} b^2 \quad \forall a, b \in \mathbb{R}, \theta \in (0, 1), \quad (2.2.1)$$

$$(a + b)^2 \leq (1 + \theta^2) a^2 + (1 + \theta^{-2}) b^2 \quad \forall a, b \in \mathbb{R}, \theta \in (0, 1). \quad (2.2.2)$$

**Lemma 2.5** *Let  $k \leq K$ . Then the estimate*

$$\| \|T_{K,k,m}v\| \|_{2,K} \lesssim \| \|v\| \|_{2,k} \quad \forall v \in V_k \quad (2.2.3)$$

*holds for  $m$  sufficiently large.*

*Proof.* Let  $v \in V_k$  be arbitrary. For any  $\omega \in (0, 1)$ , we have, from (2.1.6), (2.1.9), (2.1.8), (2.1.33), (2.1.16), (2.1.29), (2.1.36), (2.1.42) and (2.2.1) that

$$\begin{aligned}
\|T_{K,k}v\|_{2,K}^2 &= a_K(T_{K,k}v, T_{K,k}v) \\
&= a_K(R_K^m I_{K-1}^K T_{K-1,k}v, R_K^m I_{K-1}^K T_{K-1,k}v) \\
&\leq a_K(I_{K-1}^K T_{K-1,k}v, I_{K-1}^K T_{K-1,k}v) \\
&= a_{K-1}(P_K^{K-1} I_{K-1}^K T_{K-1,k}v, T_{K-1,k}v) \\
&= a_{K-1}(T_{K-1,k}v, T_{K-1,k}v) \\
&\quad + a_{K-1}((P_K^{K-1} I_{K-1}^K - Id_{K-1})T_{K-1,k}v, T_{K-1,k}v) \\
&\leq \|T_{K-1,k}v\|_{2,K-1}^2 \\
&\quad + \| (P_K^{K-1} I_{K-1}^K - Id_{K-1})T_{K-1,k}v \|_{2,K-1} \|T_{K-1,k}v\|_{2,K-1} \\
&\leq (1 + \theta_K^2) \|T_{K-1,k}v\|_{2,K-1}^2 + C\theta_K^{-2} h_K^{2\alpha} \|T_{K-1,k}v\|_{2+\alpha,K-1}^2 \\
&\leq (1 + \theta_K^2) \|T_{K-1,k}v\|_{2,K-1}^2 + C\theta_K^{-2} (1 + \omega)^{2(K-k)} h_K^{2\alpha} \|v\|_{2+\alpha,k}^2,
\end{aligned}$$

provided  $m$  is sufficiently large. Iterating the inequality we have

$$\begin{aligned}
\|T_{K,k}v\|_{2,K}^2 &\leq \left[ \prod_{k+1 \leq q \leq K} (1 + \theta_q^2) \right] \|v\|_{2,k}^2 \\
&\quad + C \left[ \sum_{k+1 \leq p \leq K} \left( \prod_{p+1 \leq q \leq K} (1 + \theta_q^2) \right) \theta_p^{-2} (1 + \omega)^{2(p-k)} h_p^{2\alpha} \right] \|v\|_{2+\alpha,k}^2,
\end{aligned} \tag{2.2.4}$$

where  $\theta_{k+1}, \dots, \theta_K$  are arbitrary numbers in  $(0, 1)$ .

From (1.2.1) we know that  $h_p = 2^{k-p} h_k$ . By choosing

$$\omega = \left( \frac{4}{3} \right)^{\alpha/2} - 1$$

we have

$$(1 + \omega)^{2(p-k)} h_p^{2\alpha} = [(1 + \omega)^2 4^{-\alpha}]^{p-k} h_k^{2\alpha} = 3^{-\alpha(p-k)} h_k^{2\alpha}. \tag{2.2.5}$$

Let

$$\theta_q = \left( \frac{2}{3} \right)^{\alpha(q-k)/2} \tag{2.2.6}$$

for  $k + 1 \leq q \leq K$ . Combining (2.2.4)–(2.2.6) we have

$$\|T_{K,k}v\|_{2,k}^2 \leq \rho_1 \|v\|_{2,k}^2 + C\rho_1\rho_2 h_k^{2\alpha} \|v\|_{2+\alpha,k}^2, \quad (2.2.7)$$

where

$$\rho_1 = \prod_{j=1}^{\infty} \left[ 1 + \left(\frac{2}{3}\right)^{\alpha j} \right] < \infty \text{ and } \rho_2 = \sum_{j=1}^{\infty} \left(\frac{1}{2}\right)^{\alpha j} < \infty.$$

The estimate (2.2.3) follows from (2.1.32) and (2.2.7).  $\square$

Combined with (2.1.11) and (2.1.31), the lemma implies the following corollary.

**Corollary 2.6** *Let  $k \leq K$ . Then the estimate*

$$\|T_{k,K,m}v\|_{2,k} \lesssim \|v\|_{2,K} \quad \forall v \in V_K \quad (2.2.8)$$

*holds for  $m$  sufficiently large.*

**Lemma 2.7** *Let  $k \leq K$ . Then the estimate*

$$\|T_{K,k,m}v\|_{2-\alpha,K} \lesssim \|v\|_{2-\alpha,k} \quad \forall v \in V_k \quad (2.2.9)$$

*holds for  $m$  sufficiently large.*

*Proof.* Let  $v \in V_k$  be arbitrary. From (2.1.8), (2.1.9), (2.1.22), (2.1.33) and (2.2.3) we have

$$\begin{aligned} \|T_{K,k}v\|_{2-\alpha,K}^2 &= \|R_K^m I_{K-1}^K T_{K-1,k}v\|_{2-\alpha,K}^2 \\ &\leq \|I_{K-1}^K T_{K-1,k}v\|_{2-\alpha,K}^2 \\ &\leq (1 + \theta_K^2) \|T_{K-1,k}v\|_{2-\alpha,K-1}^2 + C\theta_K^{-2} h_K^{2\alpha} \|T_{K-1,k}v\|_{2,K-1}^2 \\ &\leq (1 + \theta_K^2) \|T_{K-1,k}v\|_{2-\alpha,K-1}^2 + C\theta_K^{-2} h_K^{2\alpha} \|v\|_{2,k}^2 \end{aligned}$$

for  $m$  sufficiently large and  $\theta_K \in (0, 1)$ . Iterating this inequality we have

$$\begin{aligned} \|T_{K,k}v\|_{2-\alpha,K}^2 &\leq \left[ \prod_{k+1 \leq q \leq K} (1 + \theta_q^2) \right] \|v\|_{2-\alpha,k}^2 \\ &\quad + C \left[ \sum_{k+1 \leq p \leq K} \left( \prod_{p+1 \leq q \leq K} (1 + \theta_q^2) \right) \theta_p^{-2} h_p^{2\alpha} \right] \|v\|_{2,k}^2, \end{aligned} \quad (2.2.10)$$

where  $\theta_{k+1}, \dots, \theta_K$  are arbitrary numbers in  $(0, 1)$ .

As in the proof in Lemma 2.5, if we choose

$$\theta_q = \left(\frac{1}{2}\right)^{\alpha(q-k)/2} \quad \text{for } k+1 \leq q \leq K, \quad (2.2.11)$$

then from (1.2.1) and (2.2.10) we have

$$\| \|T_{K,k}v\| \|_{2-\alpha,K}^2 \lesssim \| \|v\| \|_{2-\alpha,k}^2 + h_k^{2\alpha} \| \|v\| \|_{2,k}^2. \quad (2.2.12)$$

The lemma follows from (2.1.32) and (2.2.12).  $\square$

**Lemma 2.8** *Let  $k \leq K$ . Then the estimate*

$$\| \|T_{k,K,m}v\| \|_{2-\alpha,k}^2 \lesssim \| \|v\| \|_{2-\alpha,K}^2 + h_k^{2\alpha} \| \|v\| \|_{2,K}^2 \quad \forall v \in V_K \quad (2.2.13)$$

*holds for  $m$  sufficiently large.*

*Proof.* Let  $v \in V_K$  be arbitrary. From (2.1.7), (2.1.25), (2.1.33) and (2.2.3) we have

$$\begin{aligned} \| \|T_{k,K}v\| \|_{2-\alpha,k}^2 &= \| \|P_{k+1}^k R_{k+1}^m T_{k+1,K}v\| \|_{2-\alpha,k}^2 \\ &\leq (1 + \theta_{k+1}^2) \| \|T_{k+1,K}v\| \|_{2-\alpha,k+1}^2 + C\theta_{k+1}^{-2} h_{k+1}^{2\alpha} \| \|T_{k+1,K}v\| \|_{2,k+1}^2 \\ &\leq (1 + \theta_{k+1}^2) \| \|T_{k+1,K}v\| \|_{2-\alpha,k+1}^2 + C\theta_{k+1}^{-2} h_{k+1}^{2\alpha} \| \|v\| \|_{2,K}^2 \end{aligned}$$

for  $m$  sufficiently large, where  $\theta_{k+1} \in (0, 1)$  is arbitrary. Iterating the inequality we have

$$\begin{aligned} \| \|T_{k,K}v\| \|_{2-\alpha,k}^2 &\leq \left[ \prod_{k+1 \leq q \leq K} (1 + \theta_q^2) \right] \| \|v\| \|_{2-\alpha,K}^2 \\ &\quad + C \left[ \sum_{k+1 \leq p \leq K} \left( \prod_{k+1 \leq q \leq p-1} (1 + \theta_q^2) \right) \theta_p^{-2} h_p^{2\alpha} \right] \| \|v\| \|_{2,k}^2. \end{aligned} \quad (2.2.14)$$

Choosing  $\theta_q$  by (2.2.11), we can deduce (2.2.13) from (1.2.1) and (2.2.14) as in the proof of Lemma 2.5.  $\square$

**Lemma 2.9** *Let  $k \leq K$ . Then the estimate*

$$\|T_{k,K,m}T_{K,k,m}v\|_{2-\alpha,k} \lesssim \|v\|_{2-\alpha,k} \quad \forall v \in V_k \quad (2.2.15)$$

*holds for  $m$  sufficiently large.*

*Proof.* From (2.1.32) and Lemmas 2.5, 2.7 and 2.8 we have

$$\begin{aligned} \|T_{k,K,m}T_{K,k,m}v\|_{2-\alpha,k}^2 &\lesssim \|T_{K,k,m}v\|_{2-\alpha,K}^2 + h_k^{2\alpha} \|T_{K,k,m}v\|_{2,K}^2 \\ &\lesssim \|v\|_{2-\alpha,k}^2 + h_k^{2\alpha} \|v\|_{2,k}^2 \lesssim \|v\|_{2-\alpha,k}^2, \end{aligned}$$

provided  $m$  is sufficiently large. □

We are ready to establish the main result of this section.

**Lemma 2.10** (*Strengthened Cauchy-Schwarz Inequality with Smoothing*)

*Let  $1 \leq j \leq k \leq K$ . Given any  $\omega \in (0, 1)$ , the estimate*

$$\begin{aligned} a_K(T_{K,j,m}R_j^q v_j, T_{K,k,m}R_k^q v_k) & \quad (2.2.16) \\ &\lesssim q^{-\alpha/2} \left(\frac{1+\omega}{2^\alpha}\right)^{k-j} (h_j^{-\alpha} \|v_j\|_{2-\alpha,k}) (h_k^{-\alpha} \|v_k\|_{2-\alpha,k}) \end{aligned}$$

*holds for all  $v_j \in V_j$  and  $v_k \in V_k$ , provided  $m$  is sufficiently large.*

*Proof.* Let  $v_j \in V_j$  and  $v_k \in V_k$  be arbitrary. Given any  $\omega \in (0, 1)$ , from (1.2.1), (2.1.12), (2.1.11), (2.1.31), (2.1.33), (2.1.34), (2.1.41) and Lemma 2.9 we have

$$\begin{aligned} a_K(T_{K,j}R_j^q v_j, T_{K,k}R_k^q v_k) &= a_j(R_j^q v_j, T_{j,k}T_{k,K}T_{K,k}R_k^q v_k) \\ &\leq \|R_j^q v_j\|_{2+\alpha,j} \|T_{j,k}T_{k,K}T_{K,k}R_k^q v_k\|_{2-\alpha,j} \\ &\lesssim \frac{h_j^{-2\alpha}}{q^{\alpha/2}} \|v_j\|_{2-\alpha,j} (1+\omega)^{k-j} \|v_k\|_{2-\alpha,k} \\ &= \frac{(1+\omega)^{k-j}}{q^{\alpha/2}} \left(\frac{h_k}{h_j}\right)^\alpha (h_j^{-\alpha} \|v_j\|_{2-\alpha,j}) (h_k^{-\alpha} \|v_k\|_{2-\alpha,k}) \\ &= q^{-\alpha/2} \left(\frac{1+\omega}{2^\alpha}\right)^{k-j} (h_j^{-\alpha} \|v_j\|_{2-\alpha,j}) (h_k^{-\alpha} \|v_k\|_{2-\alpha,k}), \end{aligned}$$

provided  $m$  is sufficiently large. □

**Corollary 2.11** *Let  $v_k \in V_k$  for  $1 \leq k \leq K$ . Then the estimate*

$$a_K \left( \sum_{k=1}^K T_{K,k,m} R_k^q v_k, \sum_{k=1}^K T_{K,k,m} R_k^q v_k \right) \lesssim q^{-\alpha/2} \sum_{k=1}^K h_k^{-2\alpha} \|v_k\|_{2-\alpha,k}^2 \quad (2.2.17)$$

*holds for  $m$  sufficiently large.*

*Proof.* Given any  $\omega \in (0, 1)$ . It follows from Lemma 2.10 that

$$\begin{aligned} a_K \left( \sum_{k=1}^K T_{K,k} R_k^q v_k, \sum_{k=1}^K T_{K,k} R_k^q v_k \right) &= \sum_{j,k=1}^K a_K(T_{K,j} R_k^q v_j, T_{K,k} R_k^q v_k) \quad (2.2.18) \\ &\lesssim q^{-\alpha/2} \sum_{j,k=1}^K \left( \frac{1+\omega}{2^\alpha} \right)^{|k-j|} (h_j^{-\alpha} \|v_j\|_{2-\alpha,j}) (h_k^{-\alpha} \|v_k\|_{2-\alpha,k}) \end{aligned}$$

for  $m$  sufficiently large. We choose  $\omega$  so that  $(1+\omega)2^{-\alpha} < 1$ . The estimate (2.2.17) follows from (2.2.18) and the following discrete Young's inequality (cf. [38]):

$$\sum_{j=-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} a_{j-k} b_k \right)^2 \leq \left( \sum_{k=-\infty}^{\infty} b_k \right) \left( \sum_{j=-\infty}^{\infty} a_j^2 \right),$$

where  $a_j$  and  $b_j$  are nonnegative for  $-\infty < j < \infty$ . □

## 2.3 Convergence of V-cycle and F-cycle Algorithms

In this section we will establish the asymptotic behavior of the contraction numbers of V-cycle and F-cycle algorithms with respect to the number  $m$  of smoothing steps.

We begin with introducing an auxiliary operator  $\mathcal{E}_{K,m} : V_K \rightarrow V_K$  defined by

$$\begin{aligned} \mathcal{E}_{K,m} &= \sum_{k=2}^K T_{K,k,m} [R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) h_k^{-2\alpha} A_k^{-\alpha/2} \quad (2.3.1) \\ &\quad \times (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m] T_{k,K,m}. \end{aligned}$$

**Lemma 2.12** *The estimate*

$$\|\mathcal{E}_{K,m} v\|_{2,K} \lesssim m^{\alpha/2} \|v\|_{2,K} \quad \forall v \in V_K \quad (2.3.2)$$

*holds for  $m$  sufficiently large.*

*Proof.* Let  $v \in V_K$  be arbitrary and let

$$v_k = (Id_k - I_{k-1}^k P_k^{k-1}) h_k^{-2\alpha} A_k^{-\alpha/2} (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m T_{k,K,m} v. \quad (2.3.3)$$

From (2.2.17), (2.3.1) and (2.3.3) we have

$$\begin{aligned} a_K(\mathcal{E}_{K,m} v, \mathcal{E}_{K,m} v) &= a_K \left( \sum_{k=2}^K T_{K,k,m} R_k^m v_k, \sum_{k=2}^K T_{K,k,m} R_k^m v_k \right) \\ &\lesssim m^{-\alpha/2} \sum_{k=2}^K h_k^{-2\alpha} \|v_k\|_{2-\alpha,k}^2 \end{aligned} \quad (2.3.4)$$

for  $m$  sufficiently large. From (2.1.24), (2.1.30) and (2.3.3) we have

$$\begin{aligned} \|v_k\|_{2-\alpha,k} &\lesssim \|A_k^{-\alpha/2} (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m T_{k,K,m} v\|_{2+\alpha,k} \\ &= \|A_k^{-\alpha/4} (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m T_{k,K,m} v\|_{2,k}. \end{aligned} \quad (2.3.5)$$

Let  $w_k = (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m T_{k,K,m} v$ . From (2.1.6), (2.1.10), (2.1.11), (2.1.16), (2.1.26) and (2.3.1) we have

$$\begin{aligned} &\sum_{k=2}^K h_k^{-2\alpha} \|A_k^{-\alpha/4} (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m T_{k,K,m} v\|_{2,k}^2 \\ &= \sum_{k=2}^K h_k^{-2\alpha} a_k(A_k^{-\alpha/4} w_k, A_k^{-\alpha/4} (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m T_{k,K,m} v) \\ &= \sum_{k=2}^K a_K(T_{K,k,m} R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) h_k^{-2\alpha} A_k^{-\alpha/2} w_k, v) \\ &= a_K(\mathcal{E}_{K,m} v, v). \end{aligned} \quad (2.3.6)$$

Combining (2.1.29), (2.3.4)–(2.3.6) and the Cauchy-Schwarz inequality we have

$$\|\mathcal{E}_{K,m} v\|_{2,K}^2 = a_K(\mathcal{E}_{K,m} v, \mathcal{E}_{K,m} v) \lesssim m^{-\alpha/2} a_K(\mathcal{E}_{K,m} v, v) \leq m^{-\alpha/2} \|\mathcal{E}_{K,m} v\|_{2,K} \|v\|_{2,K}$$

and the lemma follows.  $\square$

We are ready to prove the convergence of the symmetric V-cycle algorithm.

**Theorem 2.13** (*Convergence of the Symmetric V-cycle Algorithm*) Let  $\mathbb{E}_{k,m}$  be the error propagation operator for the V-cycle algorithm defined by (2.1.1). Then there exist positive mesh-independent constants  $C$  and  $m_*$  such that

$$\|\mathbb{E}_{K,m}v\|_{2,K} \leq \frac{C}{m^{\alpha/2}} \|v\|_{2,K} \quad \forall v \in V_K, m \geq m_*. \quad (2.3.7)$$

*Proof.* Suppose  $K \geq 2$ . Let  $v \in V_K$  be arbitrary and

$$v_k = (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m T_{k,K,m} v. \quad (2.3.8)$$

Then from (2.1.4), (2.2.17) and (2.3.8) we have

$$\begin{aligned} a_K(\mathbb{E}_{K,m}v, \mathbb{E}_{K,m}v) &= a_K \left( \sum_{k=2}^K T_{K,k,m} R_k^m v_k, T_{K,k,m} R_k^m v_k \right) \\ &\lesssim m^{-\alpha/2} \sum_{k=2}^K h_k^{-2\alpha} \|v_k\|_{2-\alpha,k}^2 \end{aligned} \quad (2.3.9)$$

for  $m$  sufficiently large.

From (2.1.30) and (2.3.8) we have

$$\|v_k\|_{2-\alpha,k} = \|A_k^{-\alpha/4} (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m T_{k,K,m} v\|_{2,k}. \quad (2.3.10)$$

Combining (2.1.29), (2.3.6), (2.3.9) and (2.3.10) and the Cauchy-Schwarz inequality we have

$$\begin{aligned} \|\mathbb{E}_{K,m}v\|_{2,K}^2 &= a_K(\mathbb{E}_{K,m}v, \mathbb{E}_{K,m}v) \\ &\lesssim m^{-\alpha/2} a_K(\mathcal{E}_{K,m}v, v) \leq m^{-\alpha/2} \|\mathcal{E}_{K,m}v\|_{2,K} \|v\|_{2,K} \lesssim m^{-\alpha} \|v\|_{2,K}^2, \end{aligned}$$

and the theorem follows.  $\square$

**Theorem 2.14** Let  $\tilde{\mathbb{E}}_{k,m}$  be the error propagation operator for the F-cycle algorithm defined by (2.1.13). Then there exist positive mesh-independent constants  $C$  and  $m_*$  such that

$$\|\tilde{\mathbb{E}}_{k,m}v\|_{2,k} \leq \frac{C}{m^{\alpha/2}} \|v\|_{2,k} \quad \forall v \in V_k, m \geq m_*. \quad (2.3.11)$$

*Proof.* Let  $v \in V_k$  be arbitrary. Suppose the contraction number of the  $j$ th level F-cycle iteration is  $\eta_j$ , i.e.,

$$\eta_j = \inf_{v \in V_j \setminus \{0\}} \frac{\|\tilde{\mathbb{E}}_{j,m} v\|_{2,j}}{\|v\|_{2,j}} \quad (2.3.12)$$

for  $1 \leq j \leq k$ . From (2.1.15), (2.1.24), (2.1.33), (2.1.34), (2.1.35), (2.1.37), (2.3.12) and Theorem 2.13 we have

$$\begin{aligned} \|\tilde{\mathbb{E}}_{k,m} v\|_{2,k} &\leq \|R_k^m (Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{2,k} + \|I_{k-1}^k \mathbb{E}_{k-1,m} \tilde{\mathbb{E}}_{k-1,m} P_k^{k-1} R_k^m v\|_{2,k} \\ &\lesssim m^{-\alpha/4} h_k^{-\alpha} \|(Id_k - I_{k-1}^k P_k^{k-1}) R_k^m v\|_{2-\alpha,k} + m^{-\alpha/2} \|\tilde{\mathbb{E}}_{k-1,m} P_k^{k-1} R_k^m v\|_{2,k-1} \\ &\lesssim m^{-\alpha/2} h_k^\alpha \|R_k^m v\|_{2+\alpha,k} + m^{-\alpha/2} \eta_{k-1} \|P_k^{k-1} R_k^m v\|_{2,k-1} \\ &\lesssim m^{-\alpha/2} (1 + \eta_{k-1}) \|v\|_{2,k} \end{aligned}$$

for  $m \geq m_0$ , where  $m_0$  is a mesh-independent positive integer. In other words, we obtain a recursive relation about  $\eta_k$  :

$$\eta_k \leq C_0 m^{-\alpha/2} (1 + \eta_{k-1})$$

for  $m \geq m_0$ , where  $C_0$  is a mesh-independent positive constant. Since  $\eta_1 = 0$ , it follows by mathematical induction that

$$\eta_k \leq \frac{C_0}{m^{\alpha/2} - C_0}$$

for  $m \geq \max(m_0, C_0^{2/\alpha})$ . The theorem then follows.  $\square$

# Chapter 3

## Multigrid Methods for the Hsieh-Clough-Tocher Discretization

### 3.1 The Hsieh-Clough-Tocher Element

The Hsieh-Clough-Tocher macro element is defined on a triangle. The shape functions are those  $C^1$  functions on the triangle whose restrictions to each smaller triangle formed by connecting the centroid and two vertices of the triangle are cubic polynomials. The nodal variables include the evaluations of the shape functions at the vertices of the triangle, the evaluations of the gradients at the vertices and of the normal derivatives at the midpoints of the edges of the triangle (cf. Figure 3.1).

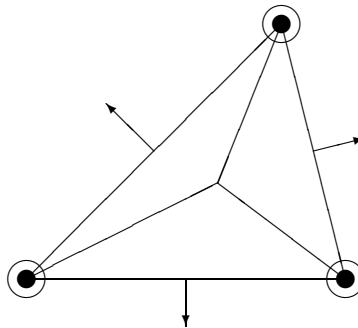


Figure 3.1: The Hsieh-Clough-Tocher macro element

Let  $\{\mathcal{T}_k\}_{k \geq 1}$  be a family of triangulations of  $\Omega$  obtained by regular subdivision, i.e.,  $\mathcal{T}_{k+1}$  is obtained by connecting the midpoints of edges of the triangles in  $\mathcal{T}_k$ . Let  $V_k$

be the Hsieh-Clough-Tocher macro element space associated with  $\mathcal{T}_k$ . Then a function  $v \in V_k$  is a function in  $C^1(\bar{\Omega})$ , whose restriction to each  $T \in \mathcal{T}_k$  is a piecewise cubic polynomial, and whose nodal values along  $\partial\Omega$  are zero. Note that  $V_k \subset H_0^2(\Omega)$  and  $V_{k-1} \not\subset V_k$ . In other words, the Hsieh-Clough-Tocher macro element is conforming but nonnested. The HCT method for the model problem is as follows:

Find  $u_k \in V_k$  such that

$$a(u_k, v) = \phi(v) \quad \forall v \in V_k. \quad (3.1.1)$$

Let  $u$  and  $u_k$  be the solutions of (1.1.2) and (3.1.1) respectively. Then it is easy to see that

$$\|u - u_k\|_a = \min_{v \in V_k} \|u - v\|_a \leq \|u - \Pi_k u\|_a, \quad (3.1.2)$$

where the operator  $\Pi_k : C^1(\bar{\Omega}) \cap H_0^2(\Omega) \rightarrow V_k$  is the nodal interpolation operator from  $C^1(\bar{\Omega})$  to  $V_k$ . Using approximation theory (cf. [31, 27]) we have

$$\|\zeta - \Pi_k \zeta\|_{L_2(\Omega)} + h_k^2 |\zeta - \Pi_k \zeta|_{H^2(\Omega)} \lesssim h_k^{2+\alpha} |\zeta|_{H^{2+\alpha}(\Omega)} \quad (3.1.3)$$

for all  $v \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega)$ .

The energy norm  $\|\cdot\|_a$  on  $H_0^2(\Omega)$  is defined by

$$\|v\|_a^2 = a(v, v) \left( = |v|_{H^2(\Omega)}^2 \right) \quad \forall v \in V_k. \quad (3.1.4)$$

Since the HCT spaces are conforming, the bilinear form  $a_k(\cdot, \cdot)$  is simply  $a(\cdot, \cdot)$ . The induced energy norms  $\|\cdot\|_{a_k}$  and  $\|\cdot\|_a$  are the same.

Combining (3.1.2) and (3.1.3) we have

$$\|u - u_k\|_a \lesssim h^\alpha \|u\|_{H^{2+\alpha}(\Omega)}. \quad (3.1.5)$$

We define the discrete inner product  $(\cdot, \cdot)_k$  on  $V_k$  by

$$\begin{aligned} (v_1, v_2)_k &:= h_k^2 \sum_{p \in \mathcal{V}_k} n(p) v_1(p) v_2(p) \\ &+ h_k^4 \sum_{p \in \mathcal{V}_k} \nabla v_1(p) \cdot \nabla v_2(p) + h_k^4 \sum_{e \in \mathcal{E}_k} \frac{\partial v_1}{\partial n}(m_e) \frac{\partial v_2}{\partial n}(m_e), \end{aligned} \quad (3.1.6)$$

where  $\mathcal{V}_k$  is the set of internal vertices of  $\mathcal{T}_k$ ,  $\mathcal{E}_k$  is the set of internal edges of  $\mathcal{T}_k$ ,  $m_e$  is the midpoint of the edge  $e$  and  $n(p) = \frac{1}{6} \times$  the number of triangles sharing the nodes  $p$  as a vertex. We have the following lemma about the discrete inner product:

**Lemma 3.1** *It holds that*

$$(v, v)_k \approx \|v\|_{L_2(\Omega)}^2 \quad \forall v \in V_k. \quad (3.1.7)$$

*Proof.* Let  $\tilde{T}$  be a triangle with  $\text{diam } \tilde{T} \approx 1$  (cf. Figure 3.2), and  $V(\tilde{T})$  be the function space of piecewise cubic polynomials that are  $C^1$  on  $\tilde{T}$ . Let  $v \in C^1(\tilde{T})$  be arbitrary. Then

$$v \mapsto \left[ \sum_{j=1}^3 (v(p_j)^2 + |\nabla v(p_j)|^2) + \sum_{j=1}^3 \left( \frac{\partial v}{\partial n}(m_j) \right)^2 \right]^{1/2}$$

defines a norm on the finite dimensional space. Therefore

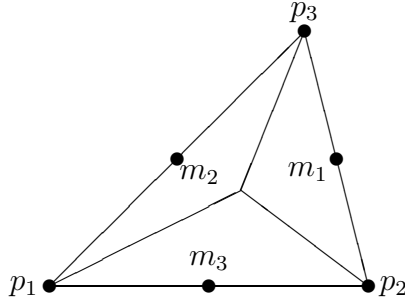


Figure 3.2: A reference triangle  $\tilde{T}$  for  $\mathcal{T}_k$

$$\sum_{j=1}^3 (v(p_j)^2 + |\nabla v(p_j)|^2) + \sum_{j=1}^3 \left( \frac{\partial v}{\partial n}(m_j) \right)^2 \approx \|v\|_{L_2(\tilde{T})}^2 \quad \forall v \in V(\tilde{T}). \quad (3.1.8)$$

By the definition (3.1.6) of the discrete inner product and using a scaling argument (cf. [27, 31]) on (3.1.8) we prove the lemma.  $\square$

We can represent the bilinear form  $a(\cdot, \cdot)$  by the operator  $A_k : V_k \longrightarrow V_k$  defined by

$$(A_k v_1, v_2)_k = a(v_1, v_2) \quad \forall v_1, v_2 \in V_k.$$

The equation (3.1.1) can then be rewritten as

$$A_k u_k = f_k, \quad (3.1.9)$$

where  $f_k \in V_k$  is defined by  $(f_k, v)_k = \phi(v)$  for all  $v \in V_k$ .

We define the coarse-to-fine intergrid transfer operator  $I_{k-1}^k : V_{k-1} \rightarrow V_k$  to be  $\Pi_k|_{V_{k-1}}$ . The fine-to-coarse operator  $I_k^{k-1} : V_k \rightarrow V_{k-1}$  is the transpose of  $I_{k-1}^k$  with respect to the discrete inner product, i.e.,

$$(I_k^{k-1} v, w)_{k-1} = (v, I_{k-1}^k w)_k \quad \forall v \in V_k, w \in V_{k-1}.$$

Now we can apply the multigrid methods in Chapter 1 to solve the equation (3.1.9).

## 3.2 Interpolation Operators and Intergrid Transfer Operators

In this section we discuss the properties of the interpolation operators and intergrid transfer operators. We begin with an estimate which follows from (3.1.3) and an interpolation of the operator  $Id - \Pi_k$  between the Sobolev spaces (cf. [7, 48]), where  $Id$  is the identity operator on  $L_2(\Omega)$ .

$$|\zeta - \Pi_k \zeta|_{H^{2-\alpha}(\Omega)} \lesssim h_k^{2\alpha} |\zeta|_{H^{2+\alpha}(\Omega)} \quad \forall \zeta \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega). \quad (3.2.1)$$

**Lemma 3.2** *Let  $\zeta \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega)$  and  $\zeta_k \in V_k$  be related by*

$$a(\zeta, v) = a(\zeta_k, v) \quad \forall v \in V_k. \quad (3.2.2)$$

*Then we have*

$$\|\zeta - \zeta_k\|_{H^{2-\alpha}(\Omega)} \lesssim h_k^{2\alpha} |\zeta|_{H^{2+\alpha}(\Omega)}. \quad (3.2.3)$$

*Proof.* Let  $\phi \in H^{-2+\alpha}(\Omega)$  be arbitrary. Then there exists  $\xi \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega)$  such that

$$a(\xi, v) = \phi(v) \quad \forall v \in H_0^2(\Omega). \quad (3.2.4)$$

From the elliptic regularity estimate (1.1.5) we have

$$\|\xi\|_{H^{2+\alpha}(\Omega)} \lesssim \|\phi\|_{H^{-2+\alpha}(\Omega)}. \quad (3.2.5)$$

Therefore since  $\Pi_k \xi \in V_k$ , and from (3.1.5), (3.2.1), (3.2.2), (3.2.4), (3.2.5) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \phi(\zeta - \zeta_k) &= a(\xi, \zeta - \zeta_k) \\ &= a(\xi - \Pi_k \xi, \zeta - \zeta_k) \\ &\leq |\xi - \Pi_k \xi|_{H^2(\Omega)} |\zeta - \zeta_k|_{H^2(\Omega)} \\ &\lesssim h_k^{2\alpha} |\xi|_{H^{2+\alpha}(\Omega)} |\zeta|_{H^{2+\alpha}(\Omega)} \\ &\lesssim h_k^{2\alpha} \|\phi\|_{H^{-2+\alpha}(\Omega)} |\zeta|_{H^{2+\alpha}(\Omega)}. \end{aligned}$$

Then by a duality formula we have

$$\|\zeta - \zeta_k\|_{H^{2-\alpha}(\Omega)} = \sup_{\phi \in H^{-2+\alpha}(\Omega) \setminus \{0\}} \frac{\phi(\zeta - \zeta_k)}{\|\phi\|_{H^{-2+\alpha}(\Omega)}} \lesssim h_k^{2\alpha} |\zeta|_{H^{2+\alpha}(\Omega)}.$$

□

Let the mesh-dependent norm  $\|\cdot\|_{s,k}$  be defined by (2.1.16). Lemma 3.1 proved the assumption (2.1.17). The assumption (2.1.18) follows from a standard inverse estimate. Therefore properties (2.1.27)–(2.1.34) in Chapter 2 are established for the HCT spaces.

The following lemma relates the mesh-dependent norms and the Sobolev norms.

**Lemma 3.3** *For  $s \in [0, 2]$  but  $s \neq \frac{1}{2}, \frac{3}{2}$  it holds that*

$$\|v\|_{s,k} \approx \|v\|_{H^s(\Omega)} \quad \forall v \in V_k. \quad (3.2.6)$$

*Proof.* Consider the identity operator  $Id_k$  on  $V_k$ . From (2.1.17) and (2.1.18) we know that it is a bounded operator from  $(V_k, \|\cdot\|_{0,k})$  into  $L_2(\Omega)$  and from  $(V_k, \|\cdot\|_{2,k})$  into  $H^2(\Omega)$ . By interpolations of Sobolev spaces and Hilbert scales (cf. [15, 39]), we have

$$\|v\|_{H^s(\Omega)} \lesssim \|v\|_{s,k} \quad \forall v \in V_k.$$

On the other hand, let  $Q_k : L_2(\Omega) \rightarrow V_k$  be the  $L_2$  projection operator on  $V_k$ , i.e., for each  $\zeta \in L_2(\Omega)$ , the function  $Q_k\zeta \in V_k$  satisfies

$$(Q_k\zeta, v)_{L_2(\Omega)} = (\zeta, v)_{L_2(\Omega)} \quad \forall v \in V_k.$$

It is known that (cf. [12])

$$\|Q_k\zeta\|_{L_2(\Omega)} \lesssim \|\zeta\|_{L_2(\Omega)} \quad \forall \zeta \in L_2(\Omega), \quad (3.2.7)$$

$$|Q_k\zeta|_{H^2(\Omega)} \lesssim |\zeta|_{H^2(\Omega)} \quad \forall \zeta \in H_0^2(\Omega). \quad (3.2.8)$$

In other words, the operator  $Q_k$  is bounded from  $L_2(\Omega)$  into  $(V_k, \|\cdot\|_{0,k})$  and from  $H_0^2(\Omega)$  into  $(V_k, \|\cdot\|_{2,k})$ . By interpolations of Sobolev spaces and Hilbert scales, we have

$$\|Q_k\zeta\|_{s,k} \lesssim \|\zeta\|_{H^s(\Omega)} \quad \forall \zeta \in H_0^s(\Omega), \quad (3.2.9)$$

for  $s \neq \frac{1}{2}, \frac{3}{2}$ . But  $Q_kv = v$  for all  $v \in V_k$ . Therefore

$$\|v\|_{s,k} \lesssim \|v\|_{H^s(\Omega)} \quad \forall v \in V_k. \quad (3.2.10)$$

□

**Lemma 3.4** For  $\zeta_k \in V_k$ , let  $\zeta \in H_0^2(\Omega)$  be defined by

$$a(\zeta, \phi) = a(\zeta_k, Q_k\phi) \quad \forall \phi \in H_0^2(\Omega). \quad (3.2.11)$$

Then

$$a(\zeta, v) = a(\zeta_k, v) \quad \forall v \in V_k, \quad (3.2.12)$$

$$|\zeta|_{H^2(\Omega)} \lesssim |||\zeta_k|||_{2,k}, \quad (3.2.13)$$

$$\|\zeta\|_{H^{2+\alpha}(\Omega)} \lesssim |||\zeta_k|||_{2+\alpha,k}, \quad (3.2.14)$$

$$\|\Pi_k \zeta\|_a \lesssim |||\zeta_k|||_{2,k}. \quad (3.2.15)$$

*Proof.* The equality (3.2.12) follows from (3.2.11) and the fact that  $Q_k v = v$  for all  $v \in V_k$ .

From (3.1.4), (3.2.8), (3.2.11) and the Cauchy-Schwarz inequality we have

$$|\zeta|_{H^2(\Omega)}^2 = a(\zeta, \zeta) = a(\zeta_k, Q_k \zeta) \leq \|\zeta_k\|_a \|Q_k \zeta\|_a \lesssim |||\zeta_k|||_{2,k} |\zeta|_{H^2(\Omega)},$$

which implies (3.2.13). From (3.2.1), (3.2.3) and an inverse estimate, we have

$$\begin{aligned} \|\Pi_k \zeta\|_a &\leq \|\Pi_k \zeta - \zeta_k\|_a + \|\zeta_k\|_a \\ &\lesssim h_k^{-\alpha} |\Pi_k \zeta - \zeta_k|_{H^{2-\alpha}(\Omega)} + \|\zeta_k\|_a \\ &\leq h_k^{-\alpha} |\Pi_k \zeta - \zeta|_{H^{2-\alpha}(\Omega)} + h_k^{-\alpha} |\zeta - \zeta_k|_{H^{2-\alpha}(\Omega)} + \|\zeta_k\|_a \\ &\lesssim \|\zeta_k\|_a, \end{aligned}$$

which proves (3.2.15).

From (2.1.31) and (3.2.9) we have

$$a(\zeta_k, Q_k \zeta) \leq |||\zeta_k|||_{2+\alpha,k} \|Q_k \zeta\|_{2-\alpha,k} \lesssim |||\zeta_k|||_{2+\alpha,k} |\zeta|_{H^{2-\alpha}(\Omega)} \quad \forall \zeta \in H_0^2(\Omega).$$

Therefore the right-hand side of (3.2.11) defines a linear functional  $\phi$  on  $H_0^2(\Omega)$  which actually belongs to  $H^{-2+\alpha}(\Omega)$  and

$$\|\phi\|_{H^{-2+\alpha}(\Omega)} \lesssim |||\zeta_k|||_{2+\alpha,k}. \quad (3.2.16)$$

The estimate (3.2.14) follows from (1.1.5) and (3.2.16).  $\square$

**Lemma 3.5** *Let  $s \in [0, 2]$ . It holds that*

$$\|\Pi_{k-1}v - v\|_{L_2(\Omega)} + h_k^s |\Pi_{k-1}v|_{H^s(\Omega)} \lesssim h_k^s |v|_{H^s(\Omega)} \quad \forall v \in V_{k-1} + V_k, \quad (3.2.17)$$

$$\|\Pi_k v - v\|_{L_2(\Omega)} + h_k^s |\Pi_k v|_{H^s(\Omega)} \lesssim h_k^s |v|_{H^s(\Omega)} \quad \forall v \in V_{k-1} + V_k. \quad (3.2.18)$$

*Proof.* Let  $T \in \mathcal{T}_{k-1}$  be divided into 4 triangles  $T_1, T_2, T_3$ , and  $T_4$  in  $\mathcal{T}_k$  and  $\tilde{T} = T/h_{k-1}$ . Then  $|\tilde{T}| \approx 1$ . (cf. Figure 3.3).

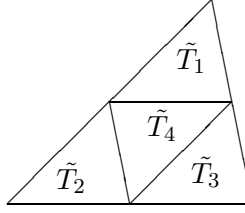


Figure 3.3: A reference triangle  $\tilde{T}$  for  $\mathcal{T}_{k-1}$

For each  $v \in V_k$ , define  $\tilde{v}(\tilde{x}) = v(h_{k-1}\tilde{x})$  for  $\tilde{x} \in \tilde{T}$ . If  $w = \Pi_{k-1}v$ , then we define  $\tilde{\Pi}_{k-1}\tilde{v}$  to be  $\tilde{w}$ .

Let  $V(\tilde{T})$  be the HCT finite element space associated with  $\tilde{T}_1, \tilde{T}_2, \tilde{T}_3$  and  $\tilde{T}_4$ , without boundary restrictions. Note that  $V(\tilde{T})$  is a finite dimensional linear space and  $|\tilde{v}|_{H^2(\tilde{T})}$  defines a norm on the quotient space  $V(\tilde{T})/P_1(\tilde{T})$ , where  $P_1(\tilde{T})$  is the space of polynomials of degree less than or equal to 1 on  $\tilde{T}$ . On the other hand,

$$v \longrightarrow \|\tilde{\Pi}_{k-1}\tilde{v} - \tilde{v}\|_{L_2(\tilde{T})}$$

defines a semi-norm on  $V(\tilde{T})/P_1(\tilde{T})$ . Therefore

$$\|\tilde{\Pi}_{k-1}\tilde{v} - \tilde{v}\|_{L_2(\tilde{T})} \lesssim |\tilde{v}|_{H^2(\tilde{T})}. \quad (3.2.19)$$

A scaling argument on (3.2.19) yields

$$\|\Pi_{k-1}v - v\|_{L_2(T)} \lesssim h_k^2 |v|_{H^2(T)} \quad \forall v \in V_k, T \in \mathcal{T}_{k-1}.$$

Therefore

$$\|\Pi_{k-1}v - v\|_{L_2(\Omega)} \lesssim h_k^2 |v|_{H^2(\Omega)} \lesssim h_k^s |v|_{H^s(\Omega)} \quad \forall v \in V_k. \quad (3.2.20)$$

From (3.2.20) and an inverse estimate, we have

$$\begin{aligned} |\Pi_{k-1}v|_{H^s(\Omega)} &\leq |\Pi_{k-1}v - v|_{H^s(\Omega)} + |v|_{H^s(\Omega)} \\ &\lesssim h_k^{-s} \|\Pi_{k-1}v - v\|_{L_2(\Omega)} + |v|_{H^s(\Omega)} \\ &\lesssim |v|_{H^s(\Omega)}. \end{aligned}$$

Therefore the estimate (3.2.17) holds for  $v \in V_k$ . The argument above also applies for the functions in a larger space  $V_{k-1} + V_k$ . This finishes the proof of (3.2.17). We can similarly prove (3.2.18).  $\square$

**Lemma 3.6** *It holds that*

$$\|\Pi_{k-1}v - v\|_{L_2(\Omega)} \lesssim h_k^{2+\alpha} \|v\|_{2+\alpha, k} \quad \forall v \in V_k, \quad (3.2.21)$$

$$\|\Pi_k v - v\|_{L_2(\Omega)} \lesssim h_k^{2+\alpha} \|v\|_{2+\alpha, k-1} \quad \forall v \in V_{k-1}. \quad (3.2.22)$$

*Proof.* Let  $\zeta_k \in V_k$  be arbitrary. We define  $\zeta \in H_0^2(\Omega)$  and  $\zeta_{k-1} \in V_{k-1}$  by (3.2.11) and

$$a(\zeta_{k-1}, v) = a(\zeta, v) \quad \forall v \in V_{k-1}. \quad (3.2.23)$$

Then from (3.1.5), Lemma 3.2, Lemma 3.4 and (3.2.17) we have

$$\begin{aligned} \|\Pi_{k-1}\zeta_k - \zeta_k\|_{L_2(\Omega)} &= \|\Pi_{k-1}(\zeta_k - \zeta_{k-1}) - (\zeta_k - \zeta_{k-1})\|_{L_2(\Omega)} \\ &\lesssim h_k^2 |\zeta_k - \zeta_{k-1}|_{H^2(\Omega)} \\ &\lesssim h_k^2 (|\zeta_k - \zeta|_{H^2(\Omega)} + |\zeta - \zeta_{k-1}|_{H^2(\Omega)}) \\ &\lesssim h_k^{2+\alpha} \|\zeta\|_{H^{2+\alpha}(\Omega)} \lesssim h_k^{2+\alpha} \|\zeta_k\|_{2+\alpha, k}, \end{aligned}$$

which proves (3.2.21). The proof of (3.2.22) is similar.  $\square$

By an inverse estimate, we have the following corollary.

**Corollary 3.7** *It holds that*

$$|\Pi_{k-1}v - v|_{H^{2-\alpha}(\Omega)} \lesssim h_k^{2\alpha} \|v\|_{2+\alpha,k} \quad \forall v \in V_k, \quad (3.2.24)$$

$$|\Pi_k v - v|_{H^{2-\alpha}(\Omega)} \lesssim h_k^{2\alpha} \|v\|_{2+\alpha,k-1} \quad \forall v \in V_{k-1}. \quad (3.2.25)$$

### 3.3 Convergence Analysis

To establish the convergence of the V-cycle and F-cycle multigrid methods, we need to verify the assumptions (2.1.17)–(2.1.24). The first two assumptions (2.1.17) and (2.1.18) have been proved in the previous section. The rest of the assumptions will be verified in this section.

**Lemma 3.8** *The estimate (2.1.21) holds. That is*

$$\|I_{k-1}^k v\|_{2,k}^2 \leq (1 + \theta^2) \|v\|_{2,k-1}^2 + C_1 \theta^{-2} h_k^{2\alpha} \|v\|_{2+\alpha,k-1}^2 \quad (3.3.1)$$

for all  $v \in V_{k-1}$  and  $\theta \in (0, 1)$ .

*Proof.* Let  $v \in V_{k-1}$  be arbitrary. Then from (2.2.2), (3.2.22) and an inverse estimate we have

$$\begin{aligned} \|I_{k-1}^k v\|_{2,k}^2 &= |\Pi_k v|_{H^2(\Omega)}^2 \\ &\leq (|v|_{H^2(\Omega)} + |\Pi_k v - v|_{H^2(\Omega)})^2 \\ &\leq (1 + \theta^2) |v|_{H^2(\Omega)}^2 + C\theta^{-2} |\Pi_k v - v|_{H^2(\Omega)}^2 \\ &\leq (1 + \theta^2) \|v\|_{2,k-1}^2 + C\theta^{-2} h_k^{-4} \|\Pi_k v - v\|_{L_2(\Omega)}^2 \\ &\leq (1 + \theta^2) \|v\|_{2,k-1}^2 + C\theta^{-2} h_k^{2\alpha} \|v\|_{2+\alpha,k-1}^2. \end{aligned}$$

□

**Lemma 3.9** *It holds that*

$$\|I_{k-1}^k v\|_{0,k}^2 \leq (1 + \theta^2) \|v\|_{0,k-1}^2 + C\theta^{-2} h_k^{2\alpha} \|v\|_{\alpha,k-1}^2 \quad (3.3.2)$$

for all  $v \in V_{k-1}$  and  $\theta \in (0, 1)$ .

*Proof.* Let  $v \in V_{k-1}$  be arbitrary and  $w = I_{k-1}^k v = \Pi_k v$ . Then,

$$\begin{aligned} w(p) &= v(p) & \forall p \in \mathcal{T}_k, \\ \nabla w(p) &= \nabla v(p) & \forall p \in \mathcal{T}_k, \\ \frac{\partial w}{\partial n}(m_e) &= \frac{\partial v}{\partial n}(m_e) & \forall e \in \mathcal{E}_k, \end{aligned}$$

where  $m_e$  is the midpoint of  $e \in \mathcal{E}_k$ . Therefore from (2.1.17) and (3.1.6) we have

$$\begin{aligned} \|v\|_{0,k-1}^2 &= h_{k-1}^2 \sum_{p \in \mathcal{V}_{k-1}} n(p) v(p)^2 \\ &+ h_{k-1}^4 \sum_{p \in \mathcal{V}_{k-1}} |\nabla v(p)|^2 + h_{k-1}^4 \sum_{e \in \mathcal{E}_{k-1}} \left[ \frac{\partial v}{\partial n}(m_e) \right]^2 \end{aligned} \quad (3.3.3)$$

and

$$\|w\|_{0,k}^2 = h_k^2 \sum_{p \in \mathcal{V}_k} n(p) v(p)^2 + h_k^4 \sum_{p \in \mathcal{V}_k} |\nabla v(p)|^2 + h_k^4 \sum_{e \in \mathcal{E}_k} \left[ \frac{\partial v}{\partial n}(m_e) \right]^2. \quad (3.3.4)$$

If  $p \in \mathcal{V}_k \setminus \mathcal{V}_{k-1}$ , then  $p$  is the midpoint of some edge  $e \in \mathcal{E}_{k-1}$ , which is the common

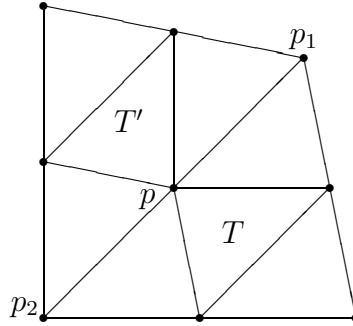


Figure 3.4: A vertex  $p \in \mathcal{T}_k \setminus \mathcal{T}_{k-1}$

edge of two triangles  $T, T' \in \mathcal{T}_{k-1}$ . Therefore  $p$  is the common vertex of 6 triangles in  $\mathcal{T}_k$  and  $n(p) = 1$ . (cf. Figure 3.4). The first part of (3.3.4) can be expressed as

$$\sum_{p \in \mathcal{V}_k} n(p)v(p)^2 = \sum_{p \in \mathcal{V}_{k-1}} n(p)v(p)^2 + \sum_{p \in \mathcal{V}_k \setminus \mathcal{V}_{k-1}} v(p)^2. \quad (3.3.5)$$

Suppose  $p_1$  and  $p_2$  are the endpoints of the edge  $e$  (cf. Figure 3.4). Then from (2.2.2) we have

$$\begin{aligned} v(p) &= [v(p_1) + (v(p) - v(p_1))]^2 \\ &\leq (1 + \theta^2)v(p_1)^2 + C\theta^{-2}[v(p) - v(p_1)]^2. \end{aligned}$$

By the Mean-Value Theorem and a standard inverse estimate we have

$$[v(p) - v(p_1)]^2 \leq |p - p_1|^2 \|\nabla v\|_{L^\infty(T)}^2 \leq C|v|_{H^1(T)}^2.$$

Then

$$v(p)^2 \leq (1 + \theta^2)v(p_1)^2 + C\theta^{-2}|v|_{H^1(T)}^2,$$

and similarly

$$v(p)^2 \leq (1 + \theta^2)v(p_2)^2 + C\theta^{-2}|v|_{H^1(T')}^2.$$

Therefore

$$v(p)^2 \leq \frac{1}{2}(1 + \theta^2)[v(p_1)^2 + v(p_2)^2] + C\theta^{-2}[|v|_{H^1(T)}^2 + |v|_{H^1(T')}^2].$$

Taking summation of the inequality above over all  $p \in \mathcal{V}_k \setminus \mathcal{V}_{k-1}$  gives

$$\begin{aligned} \sum_{p \in \mathcal{V}_k \setminus \mathcal{V}_{k-1}} v(p)^2 &\leq \frac{1}{2}(1 + \theta^2) \sum_{p \in \mathcal{V}_{k-1}} |S_p|v(p)^2 + C\theta^{-2} \sum_{T \in \mathcal{T}_{k-1}} |v|_{H^1(T)}^2 \\ &= 3(1 + \theta^2) \sum_{p \in \mathcal{V}_{k-1}} n(p)v(p)^2 + C\theta^{-2}|v|_{H^1(\Omega)}^2, \end{aligned}$$

where  $|S_p|$  is the number of triangles sharing  $p$  as a vertex. From (3.3.5) we then have

$$\sum_{p \in \mathcal{V}_k} n(p)v(p)^2 \leq 4(1 + \theta^2) \sum_{p \in \mathcal{V}_{k-1}} n(p)v(p)^2 + C\theta^{-2}|v|_{H^1(\Omega)}^2. \quad (3.3.6)$$

Let  $T \in \mathcal{T}_k$  and  $e$  be an edge of  $T$ . By a standard inverse estimate we have

$$\left[ \frac{\partial v}{\partial n}(m_e) \right]^2 \lesssim \|\nabla v\|_{L^\infty(T)}^2 \lesssim h_k^{-2} |v|_{H^1(T)}^2.$$

Therefore

$$h_k^4 \sum_{e \in \mathcal{E}_k} \left[ \frac{\partial v}{\partial n}(m_e) \right]^2 \leq C h_k^2 |v|_{H^1(\Omega)}^2. \quad (3.3.7)$$

Similarly

$$h_k^4 \sum_{p \in \mathcal{V}_k} |\nabla v(p)|^2 \leq C h_k^2 |v|_{H^1(\Omega)}^2. \quad (3.3.8)$$

From (1.2.1), (2.1.32), (3.2.6), (3.3.4), (3.3.6), (3.3.7) and (3.3.8) we have

$$\begin{aligned} \|w\|_{0,k}^2 &\leq h_k^2 \left[ 4(1 + \theta^2) \sum_{p \in \mathcal{V}_{k-1}} n(p)v(p)^2 + C\theta^{-2} |v|_{H^1(\Omega)}^2 \right] \\ &\leq (1 + \theta^2) h_{k-1}^2 \sum_{p \in \mathcal{V}_{k-1}} n(p)v(p)^2 + C\theta^{-2} h_k^2 \|v\|_{1,k-1}^2 \\ &\leq (1 + \theta^2) \|v\|_{0,k-1}^2 + C\theta^{-2} h_k^{2\alpha} \|v\|_{\alpha,k-1}^2. \end{aligned}$$

□

**Lemma 3.10** *The estimate (2.1.22) holds, i.e.,*

$$\|I_{k-1}^k v\|_{2-\alpha,k}^2 \leq (1 + \theta^2) \|v\|_{2-\alpha,k-1}^2 + C_2 \theta^{-2} h_k^{2\alpha} \|v\|_{2,k-1}^2 \quad (3.3.9)$$

for all  $v \in V_{k-1}$ ,  $\theta \in (0, 1)$ .

*Proof.* We use the approach in the proof of Lemma 6.4 in [28].

Let  $C_*$  be a constant dominating the  $C$ 's in (3.3.1) and (3.3.2). We define

$$\langle v_1, v_2 \rangle_{k-1,\theta} = (1 + \theta^2) (v_1, v_2)_{k-1} + C_* \theta^{-2} h_k^{2\alpha} (A_{k-1}^{\alpha/2} v_1, v_2)_{k-1} \quad (3.3.10)$$

for all  $v_1, v_2 \in V_{k-1}$  and  $\theta \in (0, 1)$ . Note that  $A_{k-1}$  is symmetric positive definite with respect to the inner product  $\langle \cdot, \cdot \rangle_{k-1,\theta}$ .

It follows from (3.3.1), (3.3.2) and (3.3.10) that

$$\|I_{k-1}^k v\|_{0,k}^2 \leq \langle A_{k-1}^0 v, v \rangle_{k-1,\theta} \quad \forall v \in V_{k-1},$$

$$\|I_{k-1}^k v\|_{2,k}^2 \leq \langle A_{k-1}^1 v, v \rangle_{k-1,\theta} \quad \forall v \in V_{k-1},$$

By interpolation between Hilbert scales,

$$\|I_{k-1}^k v\|_{2-\alpha,k}^2 \leq \left\langle A_{k-1}^{1-\alpha/2} v, v \right\rangle_{k-1,\theta} = (1 + \theta^2) \|v\|_{2-\alpha,k-1}^2 + C_* \theta^{-2} h_k^{2\alpha} \|v\|_{2,k-1}^2.$$

□

**Lemma 3.11** *It holds that*

$$\|\Pi_{k-1} v\|_{2,k-1}^2 \leq (1 + \theta^2) \|v\|_{2,k}^2 + C \theta^{-2} h_k^{2\alpha} \|v\|_{2+\alpha,k}^2 \quad (3.3.11)$$

for all  $v \in V_k$  and  $\theta \in (0, 1)$ .

*Proof.* Let  $v \in V_k$  be arbitrary. From (2.2.2), (2.1.18), an inverse estimate and (3.2.21) we have

$$\begin{aligned} \|\Pi_{k-1} v\|_{2,k-1}^2 &= |\Pi_{k-1} v|_{H^2(\Omega)}^2 \\ &\leq (|v|_{H^2(\Omega)} + |\Pi_{k-1} v - v|_{H^2(\Omega)})^2 \\ &\leq (\|v\|_{2,k} + C h_k^{-2} \|\Pi_{k-1} v - v\|_{L_2(\Omega)})^2 \\ &\leq (\|v\|_{2,k} + C h_k^\alpha \|v\|_{2+\alpha,k})^2 \\ &\leq (1 + \theta^2) \|v\|_{2,k}^2 + C \theta^{-2} h_k^{2\alpha} \|v\|_{2+\alpha,k}^2. \end{aligned}$$

□

**Lemma 3.12** *It holds that*

$$\|\Pi_{k-1} v\|_{0,k-1}^2 \leq (1 + \theta^2) \|v\|_{0,k}^2 + C \theta^{-2} h_k^{2\alpha} \|v\|_{\alpha,k}^2 \quad (3.3.12)$$

for all  $v \in V_k$  and  $\theta \in (0, 1)$ .

*Proof.* Let  $v \in V_k$  be arbitrary. From (3.1.6) we have

$$\|v\|_{0,k}^2 = \frac{1}{6} h_k^2 \sum_{T \in \mathcal{T}_k} \sum_{p \in \mathcal{V}_T} v(p)^2 + h_k^4 \sum_{p \in \mathcal{V}_k} |\nabla v(p)|^2 + h_k^4 \sum_{e \in \mathcal{E}_k} \left[ \frac{\partial v}{\partial n}(m_e) \right]^2, \quad (3.3.13)$$

where  $\mathcal{V}_T$  is the set of the vertices of the triangle  $T$ .

Let  $w = \Pi_{k-1} v$ . Then the nodal values of  $w$  and  $v$  on  $\mathcal{T}_{k-1}$  are the same. Therefore

$$\begin{aligned} \|w\|_{0,k-1}^2 &= \frac{1}{6} h_{k-1}^2 \sum_{T \in \mathcal{T}_{k-1}} \sum_{p \in \mathcal{V}_T} v(p)^2 + \\ &h_{k-1}^4 \sum_{p \in \mathcal{V}_{k-1}} |\nabla v(p)|^2 + h_{k-1}^4 \sum_{e \in \mathcal{E}_{k-1}} \left[ \frac{\partial v}{\partial n}(m_e) \right]^2. \end{aligned} \quad (3.3.14)$$

Let  $T \in \mathcal{T}_{k-1}$  be divided into four triangles  $T_1, T_2, T_3$  and  $T_4$  in  $\mathcal{T}_k$ , whose vertices are labeled as in Figure 3.5. Then we have

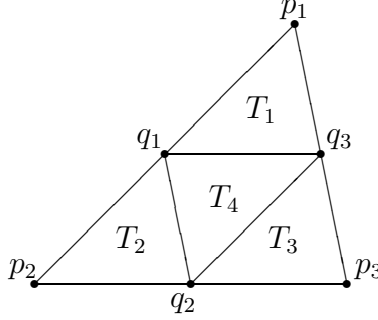


Figure 3.5: A Triangle  $T \in \mathcal{T}_{k-1}$  divided into four triangles in  $\mathcal{T}_k$

$$\begin{aligned} 4 \sum_{p \in \mathcal{V}_T} v(p)^2 &= \sum_{i=1}^3 v(p_i)^2 + 3 \sum_{i=1}^3 [v(q_i) + (v(p_i) - v(q_i))]^2 \\ &\leq \sum_{i=1}^3 v(p_i)^2 + 3 \sum_{i=1}^3 [(1 + \theta^2)v(q_i)^2 + (1 + \theta^{-2})(v(p_i) - v(q_i))^2] \quad (3.3.15) \\ &\leq (1 + \theta^2) \sum_{i=1}^4 \sum_{p \in \mathcal{V}_{T_i}} v(p)^2 + C\theta^{-2} \sum_{i=1}^4 |v|_{H^1(T_i)}^2. \end{aligned}$$

From (1.2.1) and (3.3.15) we have

$$h_{k-1}^2 \sum_{p \in \mathcal{V}_T} v(p)^2 \leq (1 + \theta^2) h_k^2 \sum_{i=1}^4 \sum_{p \in \mathcal{V}_{T_i}} v(p)^2 + C\theta^{-2} h_k^2 \sum_{i=1}^4 |v|_{H^1(T_i)}^2. \quad (3.3.16)$$

Summing up over all  $T \in \mathcal{T}_{k-1}$  gives

$$\begin{aligned} h_{k-1}^2 \sum_{T \in \mathcal{T}_{k-1}} \sum_{p \in \mathcal{V}_T} v(p)^2 & \quad (3.3.17) \\ & \leq h_k^2(1 + \theta^2) \sum_{T \in \mathcal{T}_k} \sum_{p \in \mathcal{V}_T} v(p)^2 + C\theta^{-2}h_k^2 \sum_{T \in \mathcal{T}_k} |v|_{H^1(T)}^2. \end{aligned}$$

Similar to (3.3.7) and (3.3.8), we have

$$h_{k-1}^4 \sum_{p \in \mathcal{V}_{k-1}} |\nabla v(p)|^2 + h_{k-1}^4 \sum_{e \in \mathcal{E}_{k-1}} \left[ \frac{\partial v}{\partial n}(m_e) \right]^2 \leq Ch_k^2 |v|_{H^1(\Omega)}^2. \quad (3.3.18)$$

It follows from (2.1.32), Lemma 3.3, (3.3.14), (3.3.17) and (3.3.18) that

$$\begin{aligned} \|\Pi_{k-1}v\|_{0,k-1}^2 & \leq \frac{h_k^2}{6}(1 + \theta^2) \sum_{T \in \mathcal{T}_k} \sum_{p \in \mathcal{V}_T} v(p)^2 + C\theta^{-2}h_k^2 \sum_{T \in \mathcal{T}_k} |v|_{H^1(T)}^2 \\ & \leq (1 + \theta^2) \|v\|_{0,k}^2 + C\theta^{-2}h_k^2 \|v\|_{1,k}^2 \\ & \leq (1 + \theta^2) \|v\|_{0,k}^2 + C\theta^{-2}h_k^{2\alpha} \|v\|_{\alpha,k}^2. \end{aligned}$$

□

Again, from Lemma 3.11 and Lemma 3.12, and interpolation between Hilbert scales, we have the following Lemma.

**Lemma 3.13** *It holds that*

$$\|\Pi_{k-1}v\|_{2-\alpha,k-1}^2 \leq (1 + \theta^2) \|v\|_{2-\alpha,k}^2 + C\theta^{-2}h_k^{2\alpha} \|v\|_{2,k}^2 \quad (3.3.19)$$

for all  $v \in V_k$  and  $\theta \in (0, 1)$ .

**Lemma 3.14** *Let  $\zeta_k \in V_k$ . Define  $\zeta \in H_0^2(\Omega)$  and  $\zeta_{k-1} \in V_{k-1}$  by (3.2.11) and (3.2.23). Then*

$$\|\zeta_{k-1} - P_k^{k-1}\zeta_k\|_{2-\alpha,k-1} \lesssim h_k^{2\alpha} \|\zeta_k\|_{2+\alpha,k}. \quad (3.3.20)$$

*Proof.* Let  $w \in V_{k-1}$  be arbitrary. Then from Lemma 3.4 we have

$$\begin{aligned}
a(\zeta_{k-1} - P_k^{k-1}\zeta_k, w) &= a(\zeta_{k-1}, w) - a(P_k^{k-1}\zeta_k, w) \\
&= a(\zeta, w) - a(\zeta_k, \Pi_k w) \\
&= a(\zeta, w) - a(\zeta, \Pi_k w) \\
&= a(\zeta, w - \Pi_k w).
\end{aligned}$$

From (1.1.6), (3.2.25) and Lemma 3.4 we have

$$\begin{aligned}
a(\zeta, w - \Pi_k w) &\lesssim |\zeta|_{H^{2+\alpha}(\Omega)} |w - \Pi_k w|_{H^{2-\alpha}(\Omega)} \\
&\lesssim h_k^{2\alpha} \|\zeta_k\|_{2+\alpha, k} \|w\|_{2+\alpha, k-1}.
\end{aligned}$$

Therefore

$$a(\zeta_{k-1} - P_k^{k-1}\zeta_k, w) \lesssim h_k^{2\alpha} \|\zeta_k\|_{2+\alpha, k} \|w\|_{2+\alpha, k-1}.$$

Since  $w \in V_{k-1}$  is arbitrary, it follows from (2.1.31) that

$$\|\zeta_{k-1} - P_k^{k-1}\zeta_k\|_{2-\alpha, k-1} \lesssim h_k^{2\alpha} \|\zeta_k\|_{2+\alpha, k}.$$

□

**Lemma 3.15** *The estimate (2.1.23) holds. That is*

$$\|P_k^{k-1}v\|_{2-\alpha, k-1}^2 \leq (1 + \theta^2) \|v\|_{2-\alpha, k}^2 + C\theta^{-2} h_k^{2\alpha} \|v\|_{2, k}^2 \quad (3.3.21)$$

for all  $v \in V_k$  and  $\theta \in (0, 1)$ .

*Proof.* Let  $\zeta_k \in V_k$  be arbitrary, and define  $\zeta \in H_0^2(\Omega)$  and  $\zeta_{k-1} \in V_{k-1}$  by (3.2.11) and (3.2.23). From (2.2.2) and Lemma 3.14 we have

$$\begin{aligned}
\|P_k^{k-1}\zeta_k\|_{2-\alpha, k-1}^2 &\leq (\|\zeta_{k-1}\|_{2-\alpha, k-1} + \|P_k^{k-1}\zeta_k - \zeta_{k-1}\|_{2-\alpha, k-1})^2 \\
&\leq (1 + \theta^2) \|\zeta_{k-1}\|_{2-\alpha, k-1}^2 + C\theta^{-2} \|P_k^{k-1}\zeta_k - \zeta_{k-1}\|_{2-\alpha, k-1}^2 \\
&\leq (1 + \theta^2) \|\zeta_{k-1}\|_{2-\alpha, k-1}^2 + C\theta^{-2} h_k^{4\alpha} \|\zeta_k\|_{2+\alpha, k}^2.
\end{aligned}$$

But

$$\begin{aligned} \|\zeta_{k-1}\|_{2-\alpha, k-1}^2 &\leq (\|\Pi_{k-1}\zeta_k\|_{2-\alpha, k-1} + \|\zeta_{k-1} - \Pi_{k-1}\zeta_k\|_{2-\alpha, k-1})^2 \\ &\leq (1 + \theta^2)\|\Pi_{k-1}\zeta_k\|_{2-\alpha, k-1}^2 + C\theta^{-2}\|\zeta_{k-1} - \Pi_{k-1}\zeta_k\|_{2-\alpha, k-1}^2. \end{aligned}$$

Let  $w = \zeta_{k-1} - \zeta_k \in V_{k-1} + V_k$ . Then from (3.2.17) and Lemmas 3.2, 3.3 and 3.4 we have

$$\begin{aligned} \|\zeta_{k-1} - \Pi_{k-1}\zeta_k\|_{2-\alpha, k-1} &= \|\Pi_{k-1}w\|_{2-\alpha, k-1} \\ &\approx |\Pi_{k-1}w|_{H^{2-\alpha}(\Omega)} \\ &\lesssim |w|_{H^{2-\alpha}(\Omega)} \\ &= |\zeta_{k-1} - \zeta_k|_{H^{2-\alpha}(\Omega)} \\ &\leq |\zeta_{k-1} - \zeta|_{H^{2-\alpha}(\Omega)} + |\zeta - \zeta_k|_{H^{2-\alpha}(\Omega)} \\ &\lesssim h_k^{2\alpha}|\zeta|_{H^{2+\alpha}(\Omega)} \lesssim h_k^{4\alpha}\|\zeta_k\|_{2+\alpha, k}. \end{aligned}$$

Putting these estimates together, and by Lemma 3.11 and an inverse estimate, we have

$$\begin{aligned} \|P_k^{k-1}\zeta_k\|_{2-\alpha, k-1}^2 &\leq (1 + \theta^2)^2\|\Pi_{k-1}\zeta_k\|_{2-\alpha, k-1}^2 + C\theta^{-2}h_k^{4\alpha}\|\zeta_k\|_{2+\alpha, k}^2 \\ &\leq (1 + \theta^2)^3\|\zeta_k\|_{2-\alpha, k}^2 + C\theta^{-2}h_k^{2\alpha}\|\zeta_k\|_{2, k}^2 + C\theta^{-2}h_k^{4\alpha}\|\zeta_k\|_{2+\alpha, k}^2 \\ &\leq (1 + \theta^2)^3\|\zeta_k\|_{2-\alpha, k}^2 + C\theta^{-2}h_k^{2\alpha}\|\zeta_k\|_{2, k}^2. \end{aligned}$$

The lemma follows since  $\theta \in (0, 1)$  is arbitrary.  $\square$

**Lemma 3.16** *The estimate (2.1.24) holds. That is*

$$\|(Id_k - I_{k-1}^k P_k^{k-1})v\|_{2-\alpha, k} \lesssim h_k^{2\alpha}\|v\|_{2+\alpha, k} \quad (3.3.22)$$

for all  $v \in V_k$ .

*Proof.* Let  $\zeta_k \in V_k$  be arbitrary. Define  $\zeta \in H_0^2(\Omega)$  and  $\zeta_{k-1} \in V_{k-1}$  by (3.2.11) and (3.2.23). Then from Lemma 3.3 and (3.2.18) we have

$$\begin{aligned} \|\zeta_k - I_{k-1}^k P_k^{k-1} \zeta_k\|_{2-\alpha, k} &\approx |\zeta_k - \Pi_k P_k^{k-1} \zeta_k|_{H^{2-\alpha}(\Omega)} \\ &= |\Pi_k(\zeta_k - P_k^{k-1} \zeta_k)|_{H^{2-\alpha}(\Omega)} \\ &\lesssim |\zeta_k - P_k^{k-1} \zeta_k|_{H^{2-\alpha}(\Omega)}. \end{aligned}$$

From Lemmas 3.2, 3.4, and 3.14 we have

$$\begin{aligned} &|\zeta_k - P_k^{k-1} \zeta_k|_{H^{2-\alpha}(\Omega)} \\ &\leq |\zeta_k - \zeta|_{H^{2-\alpha}(\Omega)} + |\zeta - \zeta_{k-1}|_{H^{2-\alpha}(\Omega)} + |\zeta_{k-1} - P_k^{k-1} \zeta_k|_{H^{2-\alpha}(\Omega)} \\ &\lesssim h_k^{2\alpha} \|\zeta_k\|_{2+\alpha, k}. \end{aligned}$$

The lemma then follows.  $\square$

Before verifying the last assumption, we note from (2.1.32), Lemma 3.3 and Lemma 3.15 that

$$|P_k^{k-1} v|_{H^{2-\alpha}(\Omega)} \lesssim |v|_{H^{2-\alpha}(\Omega)} \quad \forall v \in V_k. \quad (3.3.23)$$

**Lemma 3.17** *The estimate (2.1.25) holds. That is*

$$\|(Id_{k-1} - P_k^{k-1} I_{k-1}^k) v\|_{2-\alpha, k-1} \lesssim h_k^\alpha \|v\|_{2, k-1} \quad \forall v \in V_{k-1}. \quad (3.3.24)$$

*Proof.* Let  $\zeta_{k-1} \in V_{k-1}$  and define  $\zeta \in H_0^2(\Omega)$  and  $\zeta_k \in V_k$  by

$$\begin{aligned} a(\zeta, \phi) &= a(\zeta_{k-1}, Q_{k-1} \phi) \quad \forall \phi \in H_0^2(\Omega), \\ a(\zeta_k, v) &= a(\zeta, v) \quad \forall v \in V_k. \end{aligned}$$

Then from Lemma 3.3 we have

$$\begin{aligned} \|\zeta_{k-1} - P_k^{k-1} I_{k-1}^k \zeta_{k-1}\|_{2-\alpha, k-1} &\approx |\zeta_{k-1} - P_k^{k-1} I_{k-1}^k \zeta_{k-1}|_{H^{2-\alpha}(\Omega)} \\ &= |\zeta_{k-1} - P_k^{k-1} \Pi_k \zeta_{k-1}|_{H^{2-\alpha}(\Omega)}. \end{aligned} \quad (3.3.25)$$

Moreover from Lemma 3.2, Lemma 3.4, Corollary 3.7, Lemma 3.14, (3.3.23) and (3.3.25) we have

$$\begin{aligned}
& \|\zeta_{k-1} - P_k^{k-1} I_{k-1}^k \zeta_{k-1}\|_{2-\alpha, k-1} = |\zeta_{k-1} - P_k^{k-1} \Pi_k \zeta_{k-1}|_{H^{2-\alpha}(\Omega)} \\
& \leq |\zeta_{k-1} - P_k^{k-1} \zeta_k|_{H^{2-\alpha}(\Omega)} + |P_k^{k-1}(\zeta_k - \Pi_k \zeta_{k-1})|_{H^{2-\alpha}(\Omega)} \\
& \lesssim h_k^{2\alpha} |\zeta|_{H^{2+\alpha}(\Omega)} + |\zeta_k - \Pi_k \zeta_{k-1}|_{H^{2-\alpha}(\Omega)} \\
& \lesssim h_k^{2\alpha} \|\zeta_{k-1}\|_{2+\alpha, k-1} + |\zeta_k - \zeta_{k-1}|_{H^{2-\alpha}(\Omega)} + |\zeta_{k-1} - \Pi_k \zeta_{k-1}|_{H^{2-\alpha}(\Omega)} \\
& \lesssim h_k^{2\alpha} \|\zeta_{k-1}\|_{2+\alpha, k-1}.
\end{aligned}$$

□

We have verified all of the assumptions of the additive theory in Chapter 2. Theorems 2.13 and 2.14 are then established for V-cycle and F-cycle multigrid methods using the HCT element.

### 3.4 Multigrid Methods for the Reduced Hsieh-Clough-Tocher Discretization

The reduced Hsieh-Clough-Tocher element is also defined on a triangle (cf. Figure 3.1).

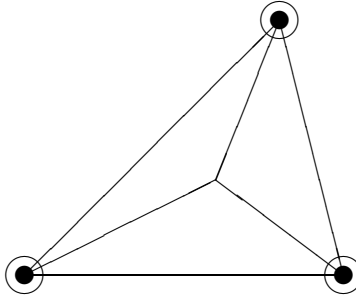


Figure 3.6: The reduced Hsieh-Clough-Tocher macro element

The shape functions of the element are those shape functions for HCT elements whose normal derivatives along the edges of triangle are linear functions. The nodal

variables include the evaluations and the gradients of the shape functions at the vertices of the triangle.

Let  $\mathcal{T}_k$  be a triangulation of the domain  $\Omega$ . Associated with  $\mathcal{T}_k$ , we can define the reduced HCT space  $\hat{V}_k$ : The functions in  $\hat{V}_k$  are those functions in  $C^1(\overline{\Omega})$  whose restrictions to each  $T \in \mathcal{T}_k$  are the reduced HCT shape functions and whose nodal values along  $\partial\Omega$  are zero. Again, the reduced HCT spaces are conforming but nonnested.

The reduced HCT method for the model problem is as follows:

Find  $u_k \in \hat{V}_k$  so that

$$a(u_k, v) = \phi(v) \quad \forall v \in \hat{V}_k. \quad (3.4.1)$$

Let  $\hat{\Pi}_k : C^1(\overline{\Omega}) \rightarrow \hat{V}_k$  be the nodal interpolation operator. It preserves quadratic polynomials. Therefore the approximation estimate (3.1.3) still holds if we replace  $\Pi_k$  by  $\hat{\Pi}_k$ . The error estimate (3.1.5) is also valid for the reduced HCT method. Moreover, if we define the discrete inner product  $(\cdot, \cdot)_k$  on  $\hat{V}_k$  by

$$(v_1, v_2)_k = h_k^2 \sum_{p \in \mathcal{V}_k} (n(p)v_1(p)v_2(p) + h_k^2 \nabla v_1(p) \cdot \nabla v_2(p)) \quad \forall v_1, v_2 \in \hat{V}_k,$$

then we have

$$(v, v)_k \approx \|v\|_{L_2(\Omega)}^2 \quad \forall v \in \hat{V}_k.$$

The intergrid transfer operator  $I_{k-1}^k : \hat{V}_{k-1} \rightarrow \hat{V}_k$  is  $\hat{\Pi}_k$  restricted on  $\hat{V}_{k-1}$ .

We can now apply the multigrid algorithms to solve (3.4.1). The convergence analysis can be carry out as we did for the HCT method.

## 3.5 Numerical Results

In this section we present some numerical results to illustrate the theorems in the previous sections.

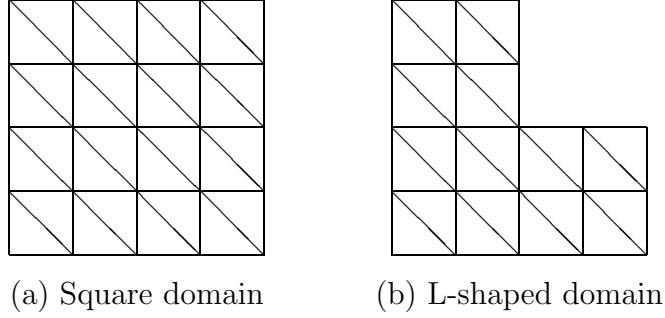


Figure 3.7: The triangulation  $\mathcal{T}_k$  for  $k = 2$ .

Our first experiment is performed on the unit square domain (cf. Figure 3.7 (a)). A family of triangulations  $\{\mathcal{T}_k\}_{k \geq 1}$  for the domain is obtained by regular subdivision. In Figure 3.7, the second level triangulation  $\mathcal{T}_2$  is shown. We compute the contraction numbers for V-cycle, F-cycle and W-cycle algorithms on different levels. The results are shown in Tables 3.1, 3.2 and 3.3.

$\gamma_{m,k,v}$	m=1	m=4	m=7	m=10	m=13	m=16	m=19	m=22
k=3	0.96	0.88	0.82	0.78	0.74	0.71	0.67	0.66
k=4	0.95	0.88	0.82	0.78	0.74	0.71	0.68	0.66
k=5	0.95	0.88	0.82	0.77	0.74	0.71	0.68	0.65
k=6	0.95	0.88	0.82	0.77	0.74	0.71	0.68	0.66
k=7	0.95	0.88	0.82	0.77	0.74	0.71	0.68	0.66
k=8	0.95	0.88	0.82	0.77	0.74	0.71	0.68	0.66

Table 3.1: HCT: Contraction numbers for V-cycle algorithms on the unit square

$\gamma_{m,k,f}$	m=1	m=4	m=7	m=10	m=13	m=16	m=19	m=22
k=3	0.96	0.88	0.82	0.76	0.73	0.70	0.68	0.65
k=4	0.96	0.88	0.82	0.77	0.74	0.71	0.68	0.65
k=5	0.96	0.88	0.81	0.77	0.74	0.71	0.68	0.65
k=6	0.96	0.88	0.81	0.77	0.74	0.70	0.68	0.65
k=7	0.96	0.88	0.81	0.77	0.74	0.70	0.68	0.65
k=8	0.96	0.88	0.81	0.77	0.74	0.71	0.68	0.65

Table 3.2: HCT: Contraction numbers for F-cycle algorithms on the unit square

$\gamma_{m,k,w}$	m=1	m=4	m=7	m=10	m=13	m=16	m=19	m=22
k=3	0.95	0.88	0.82	0.76	0.71	0.69	0.66	0.65
k=4	0.95	0.88	0.81	0.76	0.72	0.70	0.67	0.64
k=5	0.95	0.87	0.81	0.76	0.73	0.70	0.67	0.65
k=6	0.95	0.87	0.80	0.76	0.73	0.70	0.67	0.65
k=7	0.95	0.87	0.80	0.76	0.73	0.70	0.67	0.65
k=8	0.95	0.87	0.80	0.76	0.73	0.70	0.67	0.65

Table 3.3: HCT: Contraction numbers for W-cycle algorithms on the unit square

The results show that the algorithms converge for  $m$  as small as 1. We can also see from the tables that the convergence rates of the three algorithms are almost the same.

Since the domain is convex, we have full elliptic regularity, i.e., the elliptic regularity index  $\alpha = 1$ . According to Theorems 2.13 and 2.14, there exists a constant  $C$ , independent of  $k$  and  $m$ , such that

$$m^{1/2}\gamma_{k,m,v} \leq C.$$

We can see these properties from Table 3.4. It turns out that the constant  $C$  could be just 4.

$m^{1/2}\gamma_{m,k,v}$	m=10	m=20	m=30	m=40	m=50	m=60	m=70	m=80
k=3	2.42	2.97	3.24	3.28	3.40	3.35	3.49	3.44
k=4	2.45	3.02	3.29	3.27	3.29	3.27	3.39	3.29
k=5	2.45	3.01	3.26	3.26	3.31	3.30	3.21	3.13
k=6	2.44	3.00	3.25	3.25	3.31	3.26	3.15	3.09
k=7	2.45	3.00	3.25	3.25	3.31	3.24	3.14	3.06
k=8	2.44	3.01	3.24	3.24	3.31	3.23	3.13	3.05

Table 3.4: HCT: V-cycle results on the unit square

Next, we use an L-shaped domain (cf. Figure 3.7 (b)) which is a nonconvex domain. The index  $\alpha_*$  of elliptic regularity is the smallest positive solution of the following equation (cf. [35]):

$$\sin^2\left(\frac{3\pi}{2}\alpha\right) = \alpha^2 \sin^2\frac{3\pi}{2}.$$

Therefore  $\alpha_* = 0.5444837368$ . The numerical results are shown in Tables 3.5 and 3.6.

They are also consistent with the theoretical results.

$m^{\alpha_*/2}\gamma_{m,k,v}$	m=10	m=20	m=30	m=40	m=50	m=60	m=70	m=80
k=3	1.45	1.49	1.46	1.41	1.36	1.30	1.25	1.19
k=4	1.45	1.52	1.47	1.45	1.39	1.31	1.24	1.21
k=5	1.45	1.51	1.49	1.43	1.36	1.28	1.24	1.23
k=6	1.44	1.52	1.49	1.43	1.36	1.28	1.20	1.21
k=7	1.44	1.52	1.49	1.44	1.36	1.27	1.20	1.14
k=8	1.45	1.52	1.49	1.43	1.35	1.27	1.19	1.13

Table 3.5: HCT: V-cycle results on an L-shaped domain

$m^{\alpha_*/2}\gamma_{m,k,f}$	m=10	m=20	m=30	m=40	m=50	m=60	m=70	m=80
k=3	1.43	1.50	1.47	1.44	1.36	1.32	1.23	1.18
k=4	1.44	1.50	1.49	1.45	1.37	1.30	1.21	1.13
k=5	1.45	1.51	1.48	1.44	1.36	1.29	1.21	1.15
k=6	1.44	1.51	1.49	1.45	1.36	1.28	1.20	1.13
k=7	1.44	1.51	1.49	1.44	1.36	1.28	1.20	1.12
k=8	1.44	1.51	1.49	1.44	1.36	1.28	1.19	1.11

Table 3.6: HCT: F-cycle results on an L-shaped domain

Finally, we give the results for the reduced HCT method on the unit square. We only present the contraction numbers for V-cycle algorithm. The numbers for the other two cycles are almost the same.

$\gamma_{m,k,v}$	m=1	m=2	m=3	m=4	m=5	m=6	m=7
k=3	0.94	0.92	0.89	0.86	0.84	0.82	0.79
k=4	1.26	0.92	0.90	0.87	0.85	0.82	0.80
k=5	1.79	0.93	0.90	0.87	0.85	0.83	0.80
k=6	2.33	0.93	0.90	0.88	0.85	0.83	0.80
k=7	2.89	0.93	0.90	0.88	0.85	0.83	0.80
k=8	3.47	0.93	0.90	0.88	0.85	0.83	0.80

Table 3.7: RHCT: Contraction numbers for V-cycle algorithms on the unit square

# Chapter 4

## A Framework for Multigrid Methods for Nonconforming Finite Elements

### 4.1 A Framework for Multigrid Methods for Nonconforming Finite Elements

To carry out the convergence analysis for the V-cycle and F-cycle multigrid algorithms using nonconforming finite element methods, we need to verify the assumptions (2.1.17)–(2.1.25). However, the verifications of the assumptions (2.1.24) and (2.1.25) involve the relation between mesh-dependent norms and the Sobolev norms. If the finite element spaces are nonconforming, they may not be subspaces of the Sobolev spaces. Therefore the approach in Chapter 3 cannot be applied to nonconforming finite elements.

In this chapter we present a framework for the convergence of multigrid methods using nonconforming finite elements which was introduced in [28] for second order problems. The assumptions (2.1.17)–(2.1.25) will be verified in the following sections, based on this framework.

One of the key ingredients of the framework is the relation between the nonconforming finite element and a conforming relative.

Let  $(K, \mathcal{P}, \mathcal{N})$  and  $(K, \tilde{\mathcal{P}}, \tilde{\mathcal{N}})$  be two finite elements (cf. [27, 31] for the notation),

where  $K$  is the element domain,  $\mathcal{P}$  and  $\tilde{\mathcal{P}}$  are the spaces of shape functions, and  $\mathcal{N}$  and  $\tilde{\mathcal{N}}$  are the sets of nodal variables. We say  $(K, \mathcal{P}, \mathcal{N}) \preceq (K, \tilde{\mathcal{P}}, \tilde{\mathcal{N}})$  if  $\mathcal{P} \subset \tilde{\mathcal{P}}$  and  $\mathcal{N} \subset \tilde{\mathcal{N}}$  and refer to  $(K, \tilde{\mathcal{P}}, \tilde{\mathcal{N}})$  as a “relative” of  $(K, \mathcal{P}, \mathcal{N})$ .

Let  $V_k$  and  $\tilde{V}_k$  be the corresponding finite element spaces of the two elements associated with the triangulation  $\mathcal{T}_k$ . In applications,  $V_k$  is nonconforming but  $\tilde{V}_k$  is conforming. So  $\tilde{V}_k$  is called a conforming relative of  $V_k$ . We can then obtain multi-grid convergence results for  $V_k$  by exploiting its connection with  $\tilde{V}_k$  (cf. [24, 28]). The technique of using conforming relatives in the treatment of nonconforming finite elements was also used in [22, 23].

We need two operators  $E_k : V_k \longrightarrow \tilde{V}_k$  and  $F_k : \tilde{V}_k \longrightarrow V_k$  satisfying the following properties:

**Assumption 1**

$$F_k \circ E_k = Id_k, \tag{4.1.1}$$

$$\|\tilde{v} - F_k \tilde{v}\|_{L_2(\Omega)} \lesssim h_k^2 |\tilde{v}|_{H^2(\Omega)} \quad \forall \tilde{v} \in \tilde{V}_k, \tag{4.1.2}$$

$$\|v - E_k v\|_{L_2(\Omega)} \lesssim \|v\|_{a_k} \quad \forall v \in V_k. \tag{4.1.3}$$

It is very difficult to find the operators  $E_k$  and  $F_k$  satisfying equation (4.1.1) if these two elements are not relatives.

By triangle inequalities and inverse estimates, it is easy to obtain the following lemma.

**Lemma 4.1** *Under Assumption 1, it holds that*

$$\|F_k \tilde{v}\|_{L_2(\Omega)} \lesssim \|\tilde{v}\|_{L_2(\Omega)} \quad \forall \tilde{v} \in \tilde{V}_k, \tag{4.1.4}$$

$$\|F_k \tilde{v}\|_{a_k} \lesssim |\tilde{v}|_{H^2(\Omega)} \quad \forall \tilde{v} \in \tilde{V}_k, \tag{4.1.5}$$

$$\|E_k v\|_{L_2(\Omega)} \lesssim \|v\|_{L_2(\Omega)} \quad \forall v \in V_k, \tag{4.1.6}$$

$$|E_k v|_{H^2(\Omega)} \lesssim \|v\|_{a_k} \quad \forall v \in V_k. \tag{4.1.7}$$

Let  $(\cdot, \cdot)_k$  be a discrete inner product on  $V_k$  satisfying the following assumption.

**Assumption 2** *It holds for all  $v \in V_k$  that*

$$(v, v)_k \approx \|v\|_{L_2(\Omega)}^2, \quad (4.1.8)$$

$$\|v\|_{a_k} \lesssim h_k^{-2} \|v\|_{L_2(\Omega)}. \quad (4.1.9)$$

Assumption 2 implies the assumptions (2.1.17) and (2.1.18) and hence the properties (2.1.27)–(2.1.34).

**Lemma 4.2** *For  $s \in [0, 2]$  but  $s \neq \frac{1}{2}, \frac{3}{2}$ , it holds that*

$$\|v\|_{s,k} \approx |E_k v|_{H^s(\Omega)} \quad \forall v \in V_k. \quad (4.1.10)$$

*Proof.* From (2.1.28), (2.1.29), Assumption 1, (4.1.6) and (4.1.7) we know that the operator  $E_k$  is a bounded operator from  $(V_k, \|\cdot\|_{L_2(\Omega)})$  to  $(L_2(\Omega), \|\cdot\|_{L_2(\Omega)})$ , and from  $(V_k, \|\cdot\|_{a_k})$  to  $(H_0^2(\Omega), |\cdot|_{H^2(\Omega)})$ . By interpolations of Sobolev spaces and Hilbert scales, we have

$$|E_k v|_{H^s(\Omega)} \lesssim \|v\|_{s,k} \quad \forall v \in V_k.$$

Conversely, let  $Q_k : L_2(\Omega) \longrightarrow \tilde{V}_k$  be the  $L_2$  projection operator on  $\tilde{V}_k$ . We define  $J_k : L_2(\Omega) \longrightarrow V_k$  by

$$J_k = F_k \circ Q_k. \quad (4.1.11)$$

Then from (4.1.4), (4.1.5), (3.2.8) and (3.2.7) we have

$$\|J_k v\|_{0,k} \approx \|F_k Q_k v\|_{L_2(\Omega)} \lesssim \|Q_k v\|_{L_2(\Omega)} \lesssim \|v\|_{L_2(\Omega)} \quad \forall v \in L_2(\Omega), \quad (4.1.12)$$

$$\|J_k v\|_{2,k} = \|F_k Q_k v\|_{a_k} \lesssim \|Q_k v\|_{H^2(\Omega)} \lesssim |v|_{H^2(\Omega)} \quad \forall v \in H_0^2(\Omega). \quad (4.1.13)$$

An interpolation using (4.1.12) and (4.1.13) gives

$$\|J_k v\|_{s,k} \lesssim |v|_{H^s(\Omega)} \quad \forall v \in H_0^s(\Omega) \quad (4.1.14)$$

for all  $v \in H_0^s(\Omega)$  and  $s \in [0, 2]$  but  $s \neq \frac{1}{2}, \frac{3}{2}$ .

From (4.1.1), (4.1.11) and (4.1.14) we have

$$\|v\|_{s,k} = \|J_k E_k v\|_{s,k} \lesssim |E_k v|_{H_0^s(\Omega)}.$$

□

**Lemma 4.3** *It holds that*

$$\sum_{T \in \mathcal{T}_k} |v|_{H^1(T)}^2 \approx \|v\|_{1,k}^2 \quad \forall v \in V_k. \quad (4.1.15)$$

*Proof.* From (4.1.3), (4.1.10) and an inverse estimate we have

$$\begin{aligned} \sum_{T \in \mathcal{T}_k} |v|_{H^1(T)}^2 &\leq \sum_{T \in \mathcal{T}_k} (|E_k v - v|_{H^1(T)} + |E_k v|_{H^1(T)})^2 \\ &\lesssim \sum_{T \in \mathcal{T}_k} |E_k v - v|_{H^1(T)}^2 + \sum_{T \in \mathcal{T}_k} |E_k v|_{H^1(T)}^2 \\ &\lesssim h_k^{-2} \|E_k v - v\|_{L_2(\Omega)}^2 + |E_k v|_{H^1(\Omega)}^2 \\ &\lesssim h_k^2 \|v\|_{a_k}^2 + \|v\|_{1,k}^2 \lesssim \|v\|_{1,k}^2. \end{aligned}$$

Conversely from (4.1.3), (4.1.10) and an inverse estimate we have

$$\begin{aligned} \|v\|_{1,k}^2 &\lesssim |E_k v|_{H^1(\Omega)}^2 = \sum_{T \in \mathcal{T}_k} |E_k v|_{H^1(T)}^2 \\ &\leq \sum_{T \in \mathcal{T}_k} (|v - E_k v|_{H^1(T)} + |v|_{H^1(T)})^2 \\ &\lesssim \sum_{T \in \mathcal{T}_k} |v - E_k v|_{H^1(T)}^2 + \sum_{T \in \mathcal{T}_k} |v|_{H^1(T)}^2 \\ &\lesssim h_k^{-2} \|v - E_k v\|_{L_2(\Omega)}^2 + \sum_{T \in \mathcal{T}_k} |v|_{H^1(T)}^2 \\ &\lesssim h_k^2 \|v\|_{a_k}^2 + \sum_{T \in \mathcal{T}_k} |v|_{H^1(T)}^2 \\ &\lesssim \sum_{T \in \mathcal{T}_k} |v|_{H^1(T)}^2. \end{aligned}$$

□

Let  $\Pi_k : H_0^2(\Omega) \longrightarrow V_k$  be an interpolation operator that satisfies the following approximation properties:

**Assumption 3** *It holds that*

$$\|\zeta - \Pi_k \zeta\|_{L_2(\Omega)} + h_k^2 \|\Pi_k \zeta\|_{a_k} \lesssim h_k^2 |\zeta|_{H^2(\Omega)} \quad (4.1.16)$$

$$\forall \zeta \in H_0^2(\Omega),$$

$$\|\zeta - \Pi_k \zeta\|_{L_2(\Omega)} + h_k^2 \|\zeta - \Pi_k \zeta\|_{a_k} \lesssim h_k^{2+\alpha} |\zeta|_{H^{2+\alpha}(\Omega)} \quad (4.1.17)$$

$$\forall \zeta \in H^{2+\alpha}(\Omega) \cap H_0^2(\Omega).$$

We assume the operator  $E_k \Pi_k$  satisfy the following property.

**Assumption 4** *It holds that*

$$\|\zeta - E_k \Pi_k \zeta\|_{L_2(\Omega)} + h_k^2 |\zeta - E_k \Pi_k \zeta|_{H^2(\Omega)} \lesssim h_k^{2+\alpha} |\zeta|_{H^{2+\alpha}(\Omega)} \quad (4.1.18)$$

$$\forall \zeta \in H^{2+\alpha}(\Omega) \cap H_0^2(\Omega).$$

We have the following lemma.

**Lemma 4.4** *It holds that*

$$|E_k \Pi_k \zeta - \zeta|_{H^{2-\alpha}(\Omega)} \lesssim h_k^{2\alpha} |\zeta|_{H^{2+\alpha}(\Omega)} \quad \forall \zeta \in H^{2+\alpha}(\Omega) \cap H_0^2(\Omega), \quad (4.1.19)$$

$$\|E_k \Pi_k v - v\|_{L_2(\Omega)} + h_k^2 |E_k \Pi_k v|_{H^2(\Omega)} \lesssim h_k^2 |v|_{H^2(\Omega)} \quad \forall v \in H_0^2(\Omega). \quad (4.1.20)$$

*Proof.* The estimate (4.1.19) follows from (4.1.18) and an interpolation of the operator  $E_k \Pi_k - Id$  between Sobolev spaces.

Let  $v \in H_0^2(\Omega)$  be arbitrary. From (4.1.3), (4.1.6), (4.1.7), (4.1.16) and a triangle inequality we have

$$\begin{aligned} & \|E_k \Pi_k v - v\|_{L_2(\Omega)} + h_k^2 |E_k \Pi_k v|_{H^2(\Omega)} \\ & \lesssim \|E_k \Pi_k v - \Pi_k v\|_{L_2(\Omega)} + \|\Pi_k v - v\|_{L_2(\Omega)} + h_k^2 \|\Pi_k v\|_{a_k} \\ & \lesssim h_k^2 \|\Pi_k v\|_{a_k} + h_k^2 |v|_{H^2(\Omega)} \\ & \lesssim h_k^2 |v|_{H^2(\Omega)}, \end{aligned}$$

which proves the estimate (4.1.20).  $\square$

**Assumption 5** Let  $\zeta \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega)$  and  $\zeta_k \in V_k$  be related by

$$a(\zeta, E_k v) = a_k(\zeta_k, v) \quad \forall v \in V_k. \quad (4.1.21)$$

Then

$$|a_k(\zeta - \zeta_k, v)| \lesssim h_k^\alpha |\zeta|_{H^{2+\alpha}(\Omega)} \|v\|_{a_k} \quad \forall v \in V_k, \quad (4.1.22)$$

$$|a_k(\zeta - \zeta_k, \Pi_k \xi)| \lesssim h_k^{2\alpha} |\zeta|_{H^{2+\alpha}(\Omega)} |\xi|_{H^{2+\alpha}(\Omega)} \quad \forall \xi \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega). \quad (4.1.23)$$

Moreover, the operator  $\Pi_k : H_0^2(\Omega) \rightarrow V_k$  can be extended to a larger space  $H_0^2(\Omega) + V_{k-1} + V_k$ , and the following properties are satisfied.

**Assumption 6** It holds that

$$\|\Pi_{k-1} v - v\|_{L_2(\Omega)} \lesssim h_k^2 \|v\|_{a_k} \quad \forall v \in V_k.$$

**Assumption 7** It holds that

$$\|\Pi_{k-1} v\|_{a_k} \lesssim \|v\|_{a_k} \quad \forall v \in H_0^2(\Omega) + V_k.$$

**Assumption 8** It holds that

$$\|\Pi_{k-1} v\|_{0,k-1}^2 \leq (1 + \theta^2) \|v\|_{0,k}^2 + C\theta^{-2} h_k^{2\alpha} \|v\|_{\alpha,k}^2 \quad (4.1.24)$$

for all  $v \in V_k$  and  $\theta \in (0, 1)$ .

Finally we assume the intergrid transfer operators satisfy the following properties.

**Assumption 9** It holds that

$$\|I_{k-1}^k v - v\|_{L_2(\Omega)} \lesssim h_k^2 |v|_{a_{k-1}} \quad \forall v \in V_{k-1}. \quad (4.1.25)$$

**Lemma 4.5**

$$\|I_{k-1}^k v\|_{s,k} \lesssim \|v\|_{s,k-1} \quad \forall v \in V_{k-1}, 0 \leq s \leq 2, \quad (4.1.26)$$

$$\|P_k^{k-1} v\|_{t,k-1} \lesssim \|v\|_{t,k} \quad \forall v \in V_k, 2 \leq t \leq 4, \quad (4.1.27)$$

*Proof.* The estimate (4.1.25) implies that the operator  $I_{k-1}^k$  is bounded from the space  $(V_{k-1}, \|\cdot\|_{2,k-1})$  to  $(V_k, \|\cdot\|_{2,k})$  and from  $(V_{k-1}, \|\cdot\|_{2,k-1})$  to  $(V_{k1}, \|\cdot\|_{2,k})$ . Therefore it is also bounded from  $(V_{k-1}, \|\cdot\|_{s,k})$  to  $(V_k, \|\cdot\|_{s,k})$  for  $0 \leq s \leq 2$ , by an interpolation between Hilbert scales.

The estimate (4.1.27) follows from (2.1.6), (2.1.31) and (4.1.26).  $\square$

**Assumption 10** *It holds that*

$$\|I_{k-1}^k \Pi_{k-1} \zeta - \Pi_k \zeta\|_{L_2(\Omega)} \lesssim h_k^{2+\alpha} |\zeta|_{H^{2+\alpha}(\Omega)} \quad \forall \zeta \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega). \quad (4.1.28)$$

An inverse estimate implies the following lemma.

**Lemma 4.6** *The following estimates holds:*

$$\|I_{k-1}^k \Pi_{k-1} \zeta - \Pi_k \zeta\|_{2-\alpha,k} \lesssim h_k^{2\alpha} |\zeta|_{H^{2+\alpha}(\omega)} \quad \forall \zeta \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega). \quad (4.1.29)$$

**Assumption 11** *It holds that*

$$\|I_{k-1}^k v\|_{0,k}^2 \leq (1 + \theta^2) \|v\|_{0,k-1}^2 + C\theta^{-2} h_k^{2\alpha} \|v\|_{\alpha,k-1}^2 \quad (4.1.30)$$

for all  $v \in V_{k-1}$  and  $\theta \in (0, 1)$ .

## 4.2 Approximation Property

In this and next sections we will establish the convergence of the W-cycle, V-cycle and F-cycle algorithms, based on the framework in Section 4.1.

**Lemma 4.7** *Let  $\zeta \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega)$  and  $\zeta_k \in V_k$  be related by (4.1.21). Then the following estimates hold:*

$$\|\zeta - \zeta_k\|_{a_k} \lesssim h_k^\alpha |\zeta|_{H^{2+\alpha}(\Omega)}, \quad (4.2.1)$$

$$\|\|\Pi_k \zeta - \zeta_k\|\|_{2-\alpha, k} \lesssim h_k^{2\alpha} |\zeta|_{H^{2+\alpha}(\Omega)}. \quad (4.2.2)$$

*Proof.* Let  $w \in V_k$  be arbitrary. Then from (1.1.3), (1.1.4) and a triangle inequality we have

$$\begin{aligned} \|\zeta - \zeta_k\|_{a_k} &\leq \|\zeta - w\|_{a_k} + \|w - \zeta_k\|_{a_k} \\ &\lesssim \|\zeta - w\|_{a_k} + \sup_{v \in V_k \setminus \{0\}} \frac{|a_k(w - \zeta_k, v)|}{\|v\|_{a_k}} \\ &= \|\zeta - w\|_{a_k} + \sup_{v \in V_k \setminus \{0\}} \frac{|a_k(w - \zeta, v) + a_k(\zeta - \zeta_k, v)|}{\|v\|_{a_k}} \quad (4.2.3) \\ &\leq \|\zeta - w\|_{a_k} + \sup_{v \in V_k \setminus \{0\}} \frac{|a_k(w - \zeta, v)|}{\|v\|_{a_k}} + \sup_{v \in V_k \setminus \{0\}} \frac{|a_k(\zeta - \zeta_k, v)|}{\|v\|_{a_k}} \\ &\lesssim \|\zeta - w\|_{a_k} + \sup_{v \in V_k \setminus \{0\}} \frac{|a_k(\zeta - \zeta_k, v)|}{\|v\|_{a_k}}. \end{aligned}$$

From (4.1.17), (4.1.22) and (4.2.3) we have

$$\begin{aligned} \|\zeta - \zeta_k\|_{a_k} &\lesssim \inf_{w \in V_k} \|\zeta - w\|_{a_k} + \sup_{v \in V_k \setminus \{0\}} \frac{|a_k(\zeta - \zeta_k, v)|}{\|v\|_{a_k}} \\ &\lesssim \|\zeta - \Pi_k \zeta\|_{a_k} + h_k^\alpha |\zeta|_{H^{2+\alpha}(\Omega)} \\ &\lesssim h_k^\alpha |\zeta|_{H^{2+\alpha}(\Omega)}, \end{aligned}$$

which proves (4.2.1).

By (4.1.10) and duality we have

$$\begin{aligned} \|\|\Pi_k \zeta - \zeta_k\|\|_{2-\alpha, k} &\approx \|E_k(\Pi_k \zeta - \zeta_k)\|_{H^{2-\alpha}(\Omega)} \\ &= \sup_{\phi \in H^{-2+\alpha}(\Omega) \setminus \{0\}} \frac{|\phi(E_k(\Pi_k \zeta - \zeta_k))|}{\|\phi\|_{H^{-2+\alpha}(\Omega)}}. \quad (4.2.4) \end{aligned}$$

Let  $\phi \in H^{-2+\alpha}(\Omega)$  be arbitrary. We define  $\xi \in H^{2+\alpha}(\Omega)$  and  $\xi_k \in V_k$  by the following equations:

$$a(\xi, v) = \phi(v) \quad \forall v \in H_0^2(\Omega), \quad (4.2.5)$$

$$a_k(\xi_k, v) = \phi(E_k v) \quad \forall v \in V_k. \quad (4.2.6)$$

Then we have

$$\phi(E_k(\Pi_k \zeta - \zeta_k)) = a(\xi, E_k \Pi_k \zeta - \zeta) + a_k(\xi_k, \zeta - \zeta_k) + a_k(\xi - \xi_k, \zeta). \quad (4.2.7)$$

The terms on the right-hand side of (4.2.7) can be estimated as follows.

First from the elliptic regularity estimate (1.1.5) we have

$$\|\xi\|_{H^{2+\alpha}(\Omega)} \lesssim \|\phi\|_{H^{-2+\alpha}(\Omega)}. \quad (4.2.8)$$

The duality estimate (1.1.6) and (4.1.19) imply that

$$|a(\xi, E_k \Pi_k \zeta - \zeta)| \lesssim h_k^{2\alpha} |\xi|_{H^{2+\alpha}(\Omega)} |\zeta|_{H^{2+\alpha}(\Omega)}. \quad (4.2.9)$$

It follows from (4.1.17), (4.1.23) and (4.2.1) that

$$\begin{aligned} |a_k(\xi_k, \zeta - \zeta_k)| &\lesssim |a_k(\xi_k - \Pi_k \xi, \zeta - \zeta_k)| + a_k(\Pi_k \xi_k, \zeta - \zeta_k)| \\ &\lesssim h_k^{2\alpha} |\xi|_{H^{2+\alpha}(\Omega)} |\zeta|_{H^{2+\alpha}(\Omega)}. \end{aligned} \quad (4.2.10)$$

Similarly we have

$$|a_k(\xi - \xi_k, \zeta)| \lesssim h_k^{2\alpha} |\xi|_{H^{2+\alpha}(\Omega)} |\zeta|_{H^{2+\alpha}(\Omega)}. \quad (4.2.11)$$

The estimate (4.2.1) follows by combining (4.2.4) and (4.2.7)–(4.2.11).  $\square$

**Lemma 4.8** *Let  $\zeta \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega)$  be arbitrary. Let  $\zeta_k \in V_k$  and  $\zeta_{k-1} \in V_{k-1}$  be defined by (4.1.21) and*

$$a_{k-1}(\zeta_{k-1}, v) = a(\zeta, E_{k-1} v) \quad \forall v \in V_{k-1}. \quad (4.2.12)$$

Then the following estimate holds:

$$\|\zeta_{k-1} - P_k^{k-1}\zeta_k\|_{2-\alpha, k-1} \lesssim h_k^{2\alpha} |\zeta|_{H^{2+\alpha}(\Omega)}. \quad (4.2.13)$$

*Proof.* By (4.1.10) and duality we have

$$\begin{aligned} \|\zeta_{k-1} - P_k^{k-1}\zeta_k\|_{2-\alpha, k-1} &\approx |E_{k-1}(\zeta_{k-1} - P_k^{k-1}\zeta_k)|_{H^{2-\alpha}(\Omega)} \\ &= \sup_{\phi \in H^{-2+\alpha}(\Omega) \setminus \{0\}} \frac{|\phi(E_{k-1}(\zeta_{k-1} - P_k^{k-1}\zeta_k))|}{\|\phi\|_{H^{-2+\alpha}(\Omega)}}. \end{aligned} \quad (4.2.14)$$

Let  $\xi \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega)$ ,  $\xi_k \in V_k$  and  $\xi_{k-1} \in V_{k-1}$  be defined by (4.2.5), (4.2.6) and

$$a_{k-1}(\xi_{k-1}, v) = \phi(E_{k-1}v) \quad \forall v \in V_{k-1}. \quad (4.2.15)$$

Then (4.2.8) holds.

From (1.1.6), (2.1.6), (4.1.21), (4.2.12) and (4.2.15) we have

$$\begin{aligned} |\phi(E_{k-1}(\zeta_{k-1} - P_k^{k-1}\zeta_k))| &= |a_{k-1}(\zeta_{k-1} - P_k^{k-1}\zeta_k, \xi_{k-1})| \\ &= |a_{k-1}(\zeta_{k-1}, \xi_{k-1}) - a_k(\zeta_k, I_{k-1}^k \xi_{k-1})| \\ &= |a(\zeta, E_{k-1}\xi_{k-1} - E_k I_{k-1}^k \xi_{k-1})| \\ &\lesssim |\zeta|_{H^{2+\alpha}(\Omega)} |E_{k-1}\xi_{k-1} - E_k I_{k-1}^k \xi_{k-1}|_{H^{2-\alpha}(\Omega)}. \end{aligned} \quad (4.2.16)$$

From (1.2.1), (4.1.10), (4.1.19), (4.1.26), (4.1.29), (4.2.5)–(4.2.15) and Lemma 4.7 we have

$$\begin{aligned} &|E_{k-1}\xi_{k-1} - E_k I_{k-1}^k \xi_{k-1}|_{H^{2-\alpha}(\Omega)} \\ &\leq |E_{k-1}(\xi_{k-1} - \Pi_{k-1}\xi)|_{H^{2-\alpha}(\Omega)} + |E_{k-1}\Pi_{k-1}\xi - \xi|_{H^{2-\alpha}(\Omega)} \\ &\quad + |\xi - E_k \Pi_k \xi|_{H^{2-\alpha}(\Omega)} + |E_k(\Pi_k \xi - I_{k-1}^k \Pi_{k-1}\xi)|_{H^{2-\alpha}(\Omega)} \\ &\quad + |E_k I_{k-1}^k(\Pi_{k-1}\xi - \xi_{k-1})|_{H^{2-\alpha}(\Omega)} \lesssim h_k^{2\alpha} |\xi|_{H^{2-\alpha}(\Omega)}. \end{aligned} \quad (4.2.17)$$

The lemma follows from (4.2.8), (4.2.14), (4.2.16) and (4.2.17).  $\square$

We are ready to prove the following approximation property.

**Lemma 4.9** *It holds that*

$$\| (Id_k - I_{k-1}^k P_k^{k-1}) v \|_{2-\alpha, k} \lesssim h_k^{2\alpha} \| v \|_{2+\alpha, k} \quad \forall v \in V_k. \quad (4.2.18)$$

*Proof.* From (4.1.10) and duality we have

$$\| (Id_k - I_{k-1}^k P_k^{k-1}) v \|_{2-\alpha, k} \approx \sup_{\phi \in H^{-2+\alpha}(\Omega) \setminus \{0\}} \frac{|\phi(E_k(Id_k - I_{k-1}^k P_k^{k-1})v)|}{\|\phi\|_{H^{-2+\alpha}(\Omega)}}. \quad (4.2.19)$$

Let  $\phi \in H^{-2+\alpha}(\Omega)$  be arbitrary. We define  $\xi \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega)$ ,  $\xi_k \in V_k$  and  $\xi_{k-1} \in V_{k-1}$  by (4.2.5), (4.2.6) and (4.2.15). Then (4.2.8) holds.

Using (2.1.6) and (4.2.6) we have

$$\begin{aligned} & \phi(E_k(Id_k - I_{k-1}^k P_k^{k-1})v) \\ &= a_k(\xi_k, (Id_k - I_{k-1}^k P_k^{k-1})v) \\ &= a_k(\xi_k, v) - a_{k-1}(P_k^{k-1}\xi_k, P_k^{k-1}v) \\ &= a_k(\xi_k - I_{k-1}^k \xi_{k-1}, v) + a_{k-1}(\xi_{k-1} - P_k^{k-1}\xi_k, P_k^{k-1}v). \end{aligned} \quad (4.2.20)$$

The two terms on the last line of (4.2.20) can be estimated by using (1.2.1), (2.1.31), (4.1.26)–(4.1.29), and Lemmas 4.7 and 4.8 as follows:

$$\begin{aligned} a_k(\xi_k - I_{k-1}^k \xi_{k-1}, v) &\lesssim \| \xi_k - I_{k-1}^k \xi_{k-1} \|_{2-\alpha, k} \| v \|_{2+\alpha, k} \\ &\lesssim (\| \xi_k - \Pi_k \xi \|_{2-\alpha, k} + \| I_{k-1}^k \Pi_{k-1} \xi - \Pi_k \xi \|_{2-\alpha, k} \\ &\quad + \| I_{k-1}^k (\Pi_{k-1} \xi - \xi_{k-1}) \|_{2-\alpha, k}) \| v \|_{2+\alpha, k-1} \end{aligned} \quad (4.2.21)$$

$$\begin{aligned} &\lesssim h_k^{2\alpha} |\xi|_{H^{2+\alpha}(\Omega)} \| v \|_{2+\alpha, k}, \\ &|a_{k-1}(\xi_{k-1} - P_k^{k-1}\xi_k, P_k^{k-1}v)| \\ &\lesssim \| \xi_{k-1} - P_k^{k-1}\xi_k \|_{2-\alpha, k-1} \| P_k^{k-1}v \|_{2+\alpha, k-1} \\ &\lesssim h_k^{2\alpha} |\xi|_{H^{2+\alpha}(\Omega)} \| v \|_{2+\alpha, k}. \end{aligned} \quad (4.2.22)$$

The lemma follows from (4.2.8), and (4.2.19)–(4.2.22).  $\square$

The approximation property is the key estimate for the “two-grid” method. The convergence of W-cycle algorithm can then be established by a perturbation argument (cf. [5, 24, 27]).

### 4.3 Convergence Analysis

To complete the convergence analysis, we need to establish the assumptions (2.1.17)–(2.1.25). The assumptions (2.1.17) and (2.1.18) follow from Assumption 1. The approximation property (2.1.24) have been obtained in Lemma 4.9. We will prove the rest of the assumptions in this section. We begin with a lemma.

**Lemma 4.10** *Let  $J_k : L_2(\Omega) \longrightarrow V_k$  be defined by (4.1.11). Given  $\zeta_k \in V_k$ , let  $\zeta \in H_0^2(\Omega)$  be defined by*

$$a(\zeta, v) = a_k(\zeta_k, J_k v) \quad \forall v \in H_0^2(\Omega). \quad (4.3.1)$$

*Then we have*

$$a(\zeta, E_k v) = a_k(\zeta_k, v) \quad \forall v \in V_k, \quad (4.3.2)$$

$$\|\zeta\|_{H^2(\Omega)} \lesssim \|\zeta_k\|_{2,k}, \quad (4.3.3)$$

$$\|\zeta\|_{H^{2+\alpha}(\Omega)} \lesssim \|\zeta_k\|_{2+\alpha,k}, \quad (4.3.4)$$

$$\|\Pi_k \zeta\|_{a_k} \lesssim \|\zeta_k\|_{2,k}. \quad (4.3.5)$$

*Proof.* It is easy to see that

$$J_k E_k v = v \quad \forall v \in V_k. \quad (4.3.6)$$

The equality (4.3.2) follows from (4.1.11) and (4.3.6).

From (2.1.29), (4.1.13), (4.3.1) and the Cauchy-Schwarz inequality we have

$$a(\zeta, \zeta) = a_k(\zeta_k, J_k \zeta) \leq \|\zeta_k\|_{a_k} \|J_k \zeta\|_{a_k} \lesssim \|\zeta_k\|_{2,k}. \quad (4.3.7)$$

The estimate (4.3.3) follows from (2.1.31) and (4.3.7). The estimate (4.3.5) follows from (2.1.31), Assumption 7 and (4.3.3).

From (2.1.31), (4.1.14) and (4.3.1) we have

$$a_k(\zeta_k, J_k v) \leq \|\zeta_k\|_{2+\alpha, k} \|J_k v\|_{2-\alpha, k} \lesssim \|\zeta_k\|_{2+\alpha, k} |v|_{H^{2-\alpha}(\Omega)} \quad \forall v \in H_0^2(\Omega).$$

Therefore the right-hand side of (4.3.1) defines a linear functional  $\phi$  on  $H_0^2(\Omega)$  which actually belongs to  $H^{-2+\alpha}(\Omega)$  and

$$\|\phi\|_{H^{-2+\alpha}(\Omega)} \lesssim \|\zeta_k\|_{2+\alpha, k}. \quad (4.3.8)$$

The estimate (4.3.4) follows from (1.1.5) and (4.3.8).  $\square$

**Lemma 4.11** *The estimate (2.1.21) holds, i.e.,*

$$\|I_{k-1}^k v\|_{2, k}^2 \leq (1 + \theta^2) \|v\|_{2, k-1}^2 + C_1 \theta^{-2} h_k^{2\alpha} \|v\|_{2+\alpha, k-1}^2 \quad (4.3.9)$$

for all  $v \in V_{k-1}$  and  $\theta \in (0, 1)$ .

*Proof.* Let  $\zeta_{k-1} \in V_{k-1}$  be arbitrary and  $\zeta \in H_0^2(\Omega)$  be defined by

$$a(\zeta, v) = a_{k-1}(\zeta_{k-1}, J_{k-1} v) \quad \forall v \in H_0^2(\Omega). \quad (4.3.10)$$

Then from (2.2.1), (2.1.20), (2.1.29), (2.1.32), (4.2.1), (4.1.26), (4.1.17), (4.1.28), (4.3.10) and Lemma 4.10 we have

$$\begin{aligned} \|I_{k-1}^k \zeta_{k-1}\|_{2, k}^2 &\leq (\|\zeta_{k-1}\|_{a_k} + \|\zeta_{k-1} - I_{k-1}^k \zeta_{k-1}\|_{a_k})^2 \\ &\leq (1 + \theta^2) \|\zeta_{k-1}\|_{a_{k-1}}^2 + C\theta^{-2} (\|\zeta_{k-1} - \zeta\|_{a_k}^2 + \|\zeta - \Pi_k \zeta\|_{a_k}^2 \\ &\quad + \|\Pi_k \zeta - I_{k-1}^k \Pi_{k-1} \zeta\|_{a_k}^2 + \|I_{k-1}^k (\Pi_k \zeta - \zeta_k)\|_{a_k}^2) \\ &\leq (1 + \theta^2) \|\zeta_{k-1}\|_{a_{k-1}}^2 + C\theta^{-2} h_k^{2\alpha} \|\zeta_{k-1}\|_{2+\alpha, k-1}^2. \end{aligned}$$

$\square$

The following lemma can be established by interpolation between Hilbert scales using Assumption 11 and Lemma 4.11 (cf. Lemma 3.10).

**Lemma 4.12** *The estimate (2.1.22) holds, i.e.,*

$$\|I_{k-1}^k v\|_{2-\alpha,k}^2 \leq (1 + \theta^2) \|v\|_{2-\alpha,k-1}^2 + C_2 \theta^{-2} h_k^{2\alpha} \|v\|_{2,k-1}^2 \quad (4.3.11)$$

for all  $v \in V_{k-1}$  and  $\theta \in (0, 1)$ .

**Lemma 4.13** *It holds that*

$$\|\Pi_{k-1} v\|_{2,k-1}^2 \leq (1 + \theta^2) \|v\|_{2,k}^2 + C \theta^{-2} h_k^{2\alpha} \|v\|_{2+\alpha,k}^2 \quad (4.3.12)$$

for all  $v \in V_k$  and  $\theta \in (0, 1)$ .

*Proof.* Let  $\zeta_k \in V_k$  be arbitrary and define  $\zeta \in H_0^2(\Omega)$  by (4.3.1).

From (1.2.1), (2.1.29), (2.2.2), (4.2.1), (4.1.17), Assumption 7 and Lemma 4.10 we have

$$\begin{aligned} \|\Pi_{k-1} \zeta_k\|_{2,k-1}^2 &\leq (\|\zeta_k\|_{a_k} + \|\zeta_k - \Pi_{k-1} \zeta_k\|_{a_k}) \\ &\leq (1 + \theta^2) \|\zeta_k\|_{2,k}^2 + C \theta^{-2} (\|\zeta_k - \zeta\|_{a_k}^2 + \|\zeta - \Pi_{k-1} \zeta\|_{a_k}^2 \\ &\quad + \|\Pi_{k-1}(\zeta - \zeta_k)\|_{a_k}^2) \\ &\leq (1 + \theta^2) \|\zeta_k\|_{2,k}^2 + C \theta^{-2} h_k^{2\alpha} \|\zeta_k\|_{2+\alpha,k}^2. \end{aligned}$$

□

The next lemma follows from Assumption 8, Lemma 4.13 and interpolation between Hilbert scales (cf. Lemma 3.10).

**Lemma 4.14** *It holds that*

$$\|\Pi_{k-1} v\|_{2-\alpha,k-1}^2 \leq (1 + \theta^2) \|v\|_{2-\alpha,k}^2 + C \theta^{-2} h_k^{2\alpha} \|v\|_{2,k}^2 \quad (4.3.13)$$

for all  $v \in V_k$  and  $\theta \in (0, 1)$ .

**Lemma 4.15** *It holds that*

$$\|\Pi_{k-1} \zeta - \Pi_{k-1} \Pi_k \zeta\|_{L_2(\Omega)} \lesssim h_k^2 |\zeta|_{H^2(\Omega)} \quad \forall \zeta \in H_0^2(\Omega). \quad (4.3.14)$$

*Proof.* From (3.1.4), (1.2.1), (4.1.16) and Assumptions 6 and 7 we have

$$\begin{aligned}
& \|\Pi_{k-1}\zeta - \Pi_{k-1}\Pi_k\zeta\|_{L_2(\Omega)} \\
& \leq \|\Pi_{k-1}\zeta - \zeta\|_{L_2(\Omega)} + \|\zeta - \Pi_k\zeta\|_{L_2(\Omega)} + \|\Pi_k\zeta - \Pi_{k-1}\Pi_k\zeta\|_{L_2(\Omega)} \\
& \lesssim h_k^2|\zeta|_{H^2(\Omega)} + h_k^2\|\Pi_k\zeta\|_{a_k} \lesssim h_k^2|\zeta|_{H^2(\Omega)}.
\end{aligned}$$

□

**Lemma 4.16** *The estimate (2.1.23) holds, i.e.,*

$$\|P_k^{k-1}v\|_{2-\alpha,k-1}^2 \leq (1 + \theta^2)\|v\|_{2-\alpha,k}^2 + C_3\theta^{-2}h_k^{2\alpha}\|v\|_{2,k}^2 \quad (4.3.15)$$

for all  $v \in V_k$  and  $\theta \in (0, 1)$ .

*Proof.* Let  $\zeta_k \in V_k$  be arbitrary. Define  $\zeta \in H_0^2(\Omega)$  and  $\zeta_{k-1} \in V_{k-1}$  by (4.3.1) and

$$a_{k-1}(\zeta_{k-1}, v) = a(\zeta, E_{k-1}v) \quad \forall v \in V_{k-1}. \quad (4.3.16)$$

From (2.2.2) we have

$$\begin{aligned}
\|P_k^{k-1}\zeta_k\|_{2-\alpha,k-1}^2 & \leq \left( \|\Pi_{k-1}\Pi_k\zeta\|_{2-\alpha,k-1} + \|\Pi_{k-1}\Pi_k\zeta - P_k^{k-1}\zeta_k\|_{2-\alpha,k-1} \right)^2 \\
& \leq (1 + \theta^2)\|\Pi_{k-1}\Pi_k\zeta\|_{2-\alpha,k-1}^2 + C\theta^{-2}\|\Pi_{k-1}\Pi_k\zeta - P_k^{k-1}\zeta_k\|_{2-\alpha,k-1}^2.
\end{aligned} \quad (4.3.17)$$

From (2.1.29), (2.1.32), (2.2.2), (4.2.2), (4.3.13) and Lemma 4.10 we have

$$\begin{aligned}
\|\Pi_{k-1}\Pi_k\zeta\|_{2-\alpha,k-1}^2 & \leq (1 + \theta^2)\|\Pi_k\zeta\|_{2-\alpha,k}^2 + C\theta^{-2}h_k^{2\alpha}\|\Pi_k\zeta\|_{2,k}^2 \\
& \leq (1 + \theta^2) \left( \|\zeta_k\|_{2-\alpha,k} + \|\Pi_k\zeta - \zeta_k\|_{2-\alpha,k} \right)^2 + C\theta^{-2}h_k^{2\alpha}\|\zeta_k\|_{2,k}^2 \\
& \leq (1 + \theta^2)^2\|\zeta_k\|_{2-\alpha,k}^2 + C\theta^{-2}h_k^{4\alpha}|\zeta_k|_{H^{2+\alpha}(\Omega)} + C\theta^{-2}h_k^{2\alpha}\|\zeta_k\|_{2,k}^2 \\
& \leq (1 + \theta^2)^2\|\zeta_k\|_{2-\alpha,k}^2 + C\theta^{-2}h_k^{4\alpha}\|\zeta_k\|_{2+\alpha,k} + C\theta^{-2}h_k^{2\alpha}\|\zeta_k\|_{2,k}^2 \\
& \leq (1 + \theta^2)^2\|\zeta_k\|_{2-\alpha,k}^2 + C\theta^{-2}h_k^{2\alpha}\|\zeta_k\|_{2,k}^2.
\end{aligned} \quad (4.3.18)$$

From (1.2.1), (2.1.28), (2.1.32), (4.2.2), (4.3.16) and Lemmas 4.8 and 4.10 we have

$$\begin{aligned}
& \|\|\Pi_{k-1}\Pi_k\zeta - P_k^{k-1}\zeta_k\|\|_{2-\alpha,k-1}^2 \\
& \leq (\|\|\Pi_{k-1}(\Pi_k\zeta - \zeta)\|\|_{2-\alpha,k-1} + \|\|\Pi_{k-1}\zeta - \zeta_k\|\|_{2-\alpha,k-1} \\
& \quad + \|\|\zeta_k - P_k^{k-1}\zeta_k\|\|_{2-\alpha,k-1})^2 \\
& \leq C \left( h_k^{2(\alpha-2)} \|\|\Pi_{k-1}(\Pi_k\zeta - \zeta)\|\|_{0,k-1}^2 + h_k^{2\alpha} \|\|\zeta_k\|\|_{2,k}^2 \right) \\
& \leq Ch_k^{2\alpha} \|\|\zeta_k\|\|_{2,k}^2.
\end{aligned} \tag{4.3.19}$$

Combining (4.3.17)–(4.3.19) we have

$$\|P_k^{k-1}\zeta_k\|_{2-\alpha,k-1}^2 \leq (1 + \theta^2)^2 \|\|\zeta_k\|\|_{2-\alpha,k}^2 + C\theta^{-2}h_k^{2\alpha} \|\|\zeta\|\|_{2,k}^2,$$

which implies (2.1.23) since  $\theta \in (0, 1)$  is arbitrary.  $\square$

From (2.1.23) and (2.1.32) we have

$$\|P_k^{k-1}v\|_{2-\alpha,k-1} \lesssim \|v\|_{2-\alpha,k} \quad \forall v \in V_k. \tag{4.3.20}$$

**Lemma 4.17** *The estimate (2.1.25) holds, i.e.,*

$$\|(Id_{k-1} - P_k^{k-1}I_{k-1}^k)v\|_{2-\alpha,k-1} \lesssim h_k^\alpha \|v\|_{2,k-1} \quad \forall v \in V_{k-1}. \tag{4.3.21}$$

*Proof.* Let  $\zeta_{k-1} \in V_{k-1}$  be arbitrary and define  $\zeta \in H_0^2(\Omega)$  and  $\zeta_k \in V_k$  by (4.3.10) and (4.3.1) respectively.

From (2.1.31) we have

$$\begin{aligned}
& a_{k-1} \left( (Id_{k-1} - P_k^{k-1}I_{k-1}^k)\zeta_{k-1}, w \right) \\
& = a_{k-1}(\zeta_{k-1} - P_k^{k-1}\zeta_k, w) - a_{k-1} \left( P_k^{k-1}(I_{k-1}^k\zeta_{k-1} - \zeta_k), w \right) \\
& \leq \|\|\zeta_{k-1} - P_k^{k-1}\zeta_k\|\|_{2-\alpha,k-1} \|w\|_{2+\alpha,k-1} \\
& \quad + \|\|P_k^{k-1}(I_{k-1}^k\zeta_{k-1} - \zeta_k)\|\|_{2-\alpha,k-1} \|w\|_{2+\alpha,k-1}
\end{aligned}$$

for all  $w \in V_{k-1}$ .

Then from (1.2.1), (2.1.16), (2.1.32), (4.1.26), (4.1.28), (4.2.2), (4.2.13), (4.3.20) and Lemma 4.10 we have

$$\begin{aligned}
& a_{k-1} \left( (Id_{k-1} - P_k^{k-1} I_{k-1}^k) \zeta_{k-1}, w \right) \\
& \lesssim \left[ h_k^{2\alpha} |\zeta|_{H^{2+\alpha}(\Omega)} + \left\| I_{k-1}^k (\zeta_{k-1} - \Pi_{k-1} \zeta) \right\|_{2-\alpha, k} \right. \\
& \quad \left. + \left\| I_{k-1}^k \Pi_{k-1} \zeta - \Pi_k \zeta \right\|_{2-\alpha, k} + \left\| \Pi_k \zeta - \zeta_k \right\|_{2-\alpha, k-1} \right] \|w\|_{2+\alpha, k-1} \\
& \lesssim h_k^{2\alpha} |\zeta|_{H^{2+\alpha}(\Omega)} \|w\|_{2+\alpha, k-1} \\
& \lesssim h_k^{2\alpha} \left\| \zeta_{k-1} \right\|_{2+\alpha, k-1} \|w\|_{2+\alpha, k-1} \\
& \lesssim h_k^\alpha \left\| \zeta_{k-1} \right\|_{2, k-1} \|w\|_{2+\alpha, k-1}
\end{aligned}$$

for all  $w \in V_{k-1}$ , which implies (2.1.25) because of (2.1.31).  $\square$

All of the assumptions in Chapter 2 have been verified. The convergence analysis is then complete.

# Chapter 5

## Multigrid Methods Based on the Morley Element

### 5.1 Morley Finite Element Method

The Morley finite element is defined on a triangle. Its shape functions are quadratic polynomials on the triangle. Its nodal variables include the evaluations of the shape functions at the vertices of the triangles and the evaluations of the normal derivatives at the midpoints of the edges of the triangles (cf. Figure 5.1).

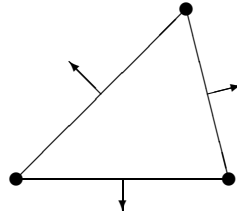


Figure 5.1: The Morley element

Let  $V_k$  be the Morley finite element space associated with  $\mathcal{T}_k$ . Then  $v \in V_k$  if and only if it has the following three properties:

1.  $v_T = v|_T$  is a quadratic polynomial for all  $T \in \mathcal{T}_k$ ,
2.  $v$  is continuous at the vertices of  $\mathcal{T}_k$  and vanishes at the vertices along  $\partial\Omega$ ,
3. The normal derivative  $\partial v/\partial n$  is continuous at the midpoints of interelement boundaries and vanishes at the midpoints along  $\partial\Omega$ .

Note that the Morley finite element spaces are nonconforming (i.e.,  $V_k \not\subset H_0^2(\Omega)$ ) and nonnested (i.e.  $V_{k-1} \not\subset V_k$ ).

The Morley method for (1.1.2) is as follows: Find  $u_k \in V_k$  such that

$$a_k(u_k, v) = \int_{\Omega} f v \, dx \quad \forall v \in V_k. \quad (5.1.1)$$

In the case where  $\phi \in H^{-2}(\Omega)$ , the modified Morley finite element method for (1.1.2) is:

Find  $u'_k \in V_k$  such that

$$a_k(u'_k, v) = \phi(E_k v) \quad \forall v \in V_k, \quad (5.1.2)$$

where  $E_k : V_k \rightarrow H_0^2(\Omega)$  will be defined in the next section.

To apply multigrid method, we need to define discrete inner product on  $V_k$  and intergrid transfer operators.

First we define a discrete inner product  $(\cdot, \cdot)_k$  on  $V_k$  by

$$(v_1, v_2)_k := h_k^2 \left[ \sum_{p \in \mathcal{V}_k} n(p) v_1(p) v_2(p) + \sum_{e \in \mathcal{E}_k} \left( \int_e \frac{\partial v_1}{\partial n} \, ds \right) \left( \int_e \frac{\partial v_2}{\partial n} \, ds \right) \right], \quad (5.1.3)$$

where  $\mathcal{V}_k$  is the set of internal vertices of  $\mathcal{T}_k$ ,  $\mathcal{E}_k$  is the set of internal edges of  $\mathcal{T}_k$  and  $n(p) = \frac{1}{6} \times$  (the number of triangles sharing the node  $p$  as a vertex). We can represent the bilinear form  $a_k(\cdot, \cdot)$  by the operator  $A_k : V_k \rightarrow V_k$  defined by

$$(A_k v_1, v_2)_k = a_k(v_1, v_2) \quad \forall v_1, v_2 \in V_k. \quad (5.1.4)$$

Then equations (5.1.1) and (5.1.2) can both be rewritten as

$$A_k u_k = f_k, \quad (5.1.5)$$

where  $f_k \in V_k$  is defined by  $(f_k, v)_k = \int_{\Omega} f v \, dx$  for all  $v \in V_k$  for the standard Morley method, and  $(f_k, v)_k = \phi(E_k v)$  for all  $v \in V_k$  for the modified Morley method.

Next we define the intergrid transfer operators. We first define  $I_{k-1}^k : V_{k-1} \rightarrow V_k$ , the coarse-to-fine intergrid transfer operator.

Let  $v \in V_{k-1}$ . We define  $I_{k-1}^k v \in V_k$  by an averaging technique as follows:

1. If  $p$  is an internal vertex of  $\mathcal{T}_k$ , then

$$(I_{k-1}^k v)(p) = \frac{1}{|S_{p,k-1}|} \sum_{T \in S_{p,k-1}} v_T(p), \quad (5.1.6)$$

where  $S_{p,k-1} := \{T \in \mathcal{T}_{k-1} : p \in \partial T\}$ .

2. If  $e$  is an internal edge of  $\mathcal{T}_k$ , which means that  $e \subset \partial T_1 \cap \partial T_2$  for some  $T_1, T_2 \in \mathcal{T}_k$ , then

$$\int_e \frac{\partial(I_{k-1}^k v)}{\partial n} ds = \frac{1}{2} \left( \int_e \frac{\partial v_{T_1}}{\partial n} ds + \int_e \frac{\partial v_{T_2}}{\partial n} ds \right). \quad (5.1.7)$$

The fine-to-coarse operator  $I_k^{k-1} : V_k \longrightarrow V_{k-1}$  is defined as follows:

$$(I_{k-1}^k v, w)_k = (v, I_k^{k-1} w)_{k-1} \quad \forall v \in V_{k-1}, w \in V_k.$$

We can now apply the algorithms given in Chapter 1 to the equation (5.1.5).

To carry out the convergence analysis of the V-cycle and F-cycle algorithms, we need to verify Assumptions 1–12 in Chapter 4. Assumption 2 is simple: The proof of (4.1.8) is similar to the one of Lemma 3.1. The estimate (4.1.9) follows from a standard inverse estimate.

Assumption 2 establishes (2.1.17) and (2.1.18). Therefore properties (2.1.27)–(2.1.34) in Chapter 2 are established for the Morley spaces.

We finish this section with the definition of the Morley interpolation operator. The rest of the assumptions of the framework in Chapter 4 will be established in the following sections.

Let  $\Pi_k : H_0^2(\Omega) \longrightarrow V_k$  be the Morley nodal interpolation operator defined as follows: For each  $\zeta \in H_0^2(\Omega)$ , the function  $\Pi_k \zeta \in V_k$  satisfies

$$(\Pi_k v)(p) = v(p) \quad \text{and} \quad \int_e \frac{\partial(\Pi_k v)}{\partial n} ds = \int_e \frac{\partial v}{\partial n} ds, \quad (5.1.8)$$

where  $p$  and  $e$  range over the internal vertices and edges of  $\mathcal{T}_k$ . By applying approximation theory (cf. [27, 31]) we have

$$\|\zeta - \Pi_k \zeta\|_{L_2(\Omega)} + h_k^2 \|\Pi_k \zeta\|_{a_k} \lesssim h_k^2 |\zeta|_{H^2(\Omega)} \quad (5.1.9)$$

$$\forall \zeta \in H_0^2(\Omega),$$

$$\|\zeta - \Pi_k \zeta\|_{L_2(\Omega)} + h_k^2 \|\zeta - \Pi_k \zeta\|_{a_k} \lesssim h_k^{2+\alpha} |\zeta|_{H^{2+\alpha}(\Omega)} \quad (5.1.10)$$

$$\forall \zeta \in H^{2+\alpha}(\Omega) \cap H_0^2(\Omega).$$

The estimates (5.1.9) and (5.1.10) establish Assumption 3 in Chapter 4 for the Morley spaces.

## 5.2 The Morley Element and its Conforming Relative

It is easy to see that the HCT element is a conforming relative of the Morley element (cf. Figure 5.2). In this chapter we denote the HCT space associated with  $\mathcal{T}_k$  as  $\tilde{V}_k$  instead of  $V_k$  (which is for the Morley space).

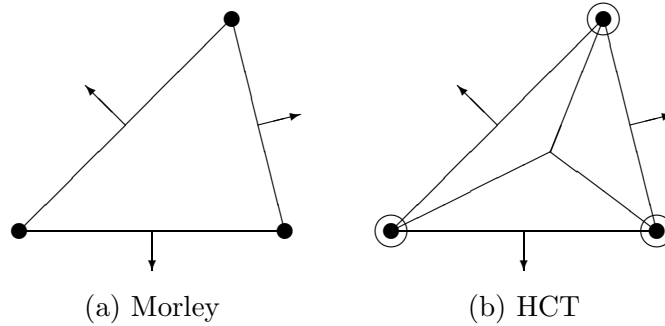


Figure 5.2: The Morley element and the HCT element

We define an operator  $E_k : V_k \longrightarrow \tilde{V}_k$ . For each  $v \in V_k$ , the function  $E_k v \in \tilde{V}_k$  is defined as follows. For any internal vertex  $p$  and internal midpoint  $m$ ,

$$(E_k v)(p) = v(p),$$

$$(\partial^\beta (E_k v))(p) = \text{average of } (\partial^\beta v_i)(p),$$

$$\frac{\partial(E_k v)}{\partial n}(m) = \frac{\partial v}{\partial n}(m),$$

where  $\beta = (0, 1)$  or  $(1, 0)$ , and  $v_i = v|_{T_i}$  for  $T_i$  with  $p$  as a vertex.

We can also define an operator  $F_k : \tilde{V}_k \rightarrow V_k$  as follows: For each  $\tilde{v} \in \tilde{V}_k$ ,  $F_k \tilde{v}$  is the function in  $V_k$  satisfying

$$(F_k \tilde{v})(p) = \tilde{v}(p) \quad \text{and} \quad \frac{\partial(F_k \tilde{v})}{\partial n}(m) = \frac{\partial \tilde{v}}{\partial n}(m)$$

for every internal vertex  $p$  and midpoint  $m$  of  $\mathcal{T}_k$ .

The operators  $E_k$  and  $F_k$  satisfy the following two properties:

**Lemma 5.1**

$$F_k \circ E_k = Id_k, \tag{5.2.1}$$

$$\|\tilde{v} - F_k \tilde{v}\|_{L_2(\Omega)} \lesssim h_k^2 |\tilde{v}|_{H^2(\Omega)} \quad \forall \tilde{v} \in \tilde{V}_k. \tag{5.2.2}$$

*Proof.* The equality (5.2.1) is obvious.

Let  $T \in \mathcal{T}_k$  and  $\tilde{T} = T/h_k$ . For each  $v \in \tilde{T}_k$ , define  $\tilde{v}(\tilde{x}) = v(h_k \tilde{x})$  for  $\tilde{x} \in \tilde{T}$ . If  $w = F_k v$ , then we define  $\tilde{F}_k \tilde{v}$  to be  $\tilde{w}$ .

Let  $V(\tilde{T}) = \{\tilde{v}_{\tilde{T}} : v \in \tilde{V}_k\}$ . Note that  $V(\tilde{T})$  is a finite dimensional linear space and  $|\tilde{v}|_{H^2(\tilde{T})}$  defines a norm on the quotient space  $V(\tilde{T})/P_1(\tilde{T})$ , where  $P_1(\tilde{T})$  is the space of polynomials of degrees less than or equal to 1 on  $\tilde{T}$ . On the other hand,

$$\tilde{v} \mapsto \|\tilde{F}_k \tilde{v} - \tilde{v}\|_{L_2(\tilde{T})}$$

defines a semi-norm on  $V(\tilde{T})/P_1(\tilde{T})$ . Therefore

$$\|\tilde{F}_k \tilde{v} - \tilde{v}\|_{L_2(\tilde{T})} \lesssim |\tilde{v}|_{H^2(\tilde{T})}. \tag{5.2.3}$$

A scaling argument on (5.2.3) yields

$$\|F_k v - v\|_{L_2(T)} \lesssim h_k^2 |v|_{H^2(T)}.$$

The estimate (5.2.2) then follows.  $\square$

A similar estimate is valid for  $E_k$ .

**Lemma 5.2** *It holds that*

$$\|v - E_k v\|_{L_2(\Omega)} \lesssim \|v\|_{a_k} \quad \forall v \in V_k. \quad (5.2.4)$$

*Proof.* Let  $v \in V_k$  and  $T \in \mathcal{T}_k$  be arbitrary. We consider the  $L_2$  norm of the function  $v - E_k v$  on  $T$ .

By an argument similar to (3.1.8) we have

$$\|v - E_k v\|_{L_2(T)}^2 \approx h_k^4 \sum_{p \in N(T)} |\nabla v_T(p) - \nabla E_k v(p)|^2, \quad (5.2.5)$$

where  $v_T$  is the function  $v$  restricted on  $T$  and  $N(T)$  is the set of three nodes of  $T$ . Let  $p$  be one of such nodes and  $p$  is shared by triangles  $T_1 (= T), T_2, \dots, T_n$  (cf. Figure 5.3). Then

$$\begin{aligned} \nabla v_T(p) - \nabla E_k v(p) &= \nabla v_{T_1}(p) - \frac{1}{n} \sum_{j=1}^n \nabla v_{T_j}(p) \\ &= \frac{1}{n} \sum_{j=2}^n (\nabla v_{T_1}(p) - \nabla v_{T_j}(p)). \end{aligned} \quad (5.2.6)$$

For  $j \geq 2$ , we have

$$\begin{aligned} \nabla v_{T_1}(p) - \nabla v_{T_j}(p) &= \sum_{i=1}^{j-1} [\nabla v_{T_i}(p) - \nabla v_{T_{i+1}}(p)] \\ &= \sum_{i=1}^{j-1} ((\nabla v_{T_i}(p) - \nabla v_{T_i}(m_i)) + (\nabla v_{T_{i+1}}(m_i) - \nabla v_{T_{i+1}}(p))). \end{aligned} \quad (5.2.7)$$

For each  $i$  we have

$$|\nabla v_{T_i}(p) - \nabla v_{T_i}(m_i)| \lesssim |v|_{H^2(T_i)}. \quad (5.2.8)$$

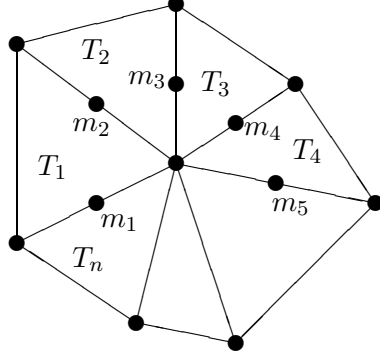


Figure 5.3: A node  $p$  shared by triangles  $T_1, T_2, \dots, T_n$ .

Combining (5.2.5)–(5.2.8) finishes the proof of the lemma.  $\square$

Lemmas 5.1 and 5.2 establish Assumption 1 of Chapter 4. Therefore the estimates (4.1.10) and (4.1.15) are valid for the Morley functions.

**Lemma 5.3** *Assumption 4 of Chapter 4 holds. That is*

$$\|\zeta - E_k \Pi_k \zeta\|_{L_2(\Omega)} + h_k^2 |\zeta - E_k \Pi_k \zeta|_{H^2(\Omega)} \lesssim h_k^{2+\alpha} |\zeta|_{H^{2+\alpha}(\Omega)} \quad (5.2.9)$$

for all  $\zeta \in H^{2+\alpha}(\Omega) \cap H_0^2(\Omega)$ .

*Proof.* Let  $V_k^*$  be the Morley space associated to  $\mathcal{T}_k$  without boundary conditions (i.e., the nodal values of the functions in  $V_k^*$  are not necessarily zero). The operator  $\Pi_k^* : H^2(\Omega) \rightarrow V_k^*$  is the nodal interpolation operator. Note that  $\Pi^*|_{H_0^2(\Omega)} = \Pi_k$ . Let  $\tilde{V}_k^*$  be the HCT space without boundary conditions and  $E_k^* : V_k^* \rightarrow H^2(\Omega)$  be the analog of  $E_k$  without boundary conditions.

Let  $T \in \mathcal{T}_k$  and  $\tilde{T} = T/h_k$ . For each  $v \in H^2(\Omega) + V_k$ , define  $\tilde{v}(\tilde{x}) = v(h_k x)$  for  $\tilde{x} \in \tilde{T}$ . For  $v \in H^2(\Omega)$ , if  $\Pi_k^* v = w$ , we define  $\tilde{\Pi}_k^* \tilde{v} = \tilde{w}$ . For  $v \in V_k$ , if  $E_k^* v = w$ , we define  $\tilde{E}_k^* \tilde{v} = \tilde{w}$ . From a triangle inequality and (5.1.9) we have

$$\|\tilde{E}_k^* \tilde{\Pi}_k^* \tilde{v}\|_{L_2(\tilde{T})} \lesssim \|\tilde{\Pi}_k^* \tilde{v}\|_{L_2(S(\tilde{T}))} \lesssim \sum_{\tilde{K} \in S(\tilde{T})} \|\tilde{v}\|_{H^2(\tilde{K})}, \quad (5.2.10)$$

where  $S(\tilde{T})$  consists of the triangle  $\tilde{T}$  and other triangles sharing at least one vertex with  $\tilde{T}$ .

Since the operator  $E_k^* \Pi_k^* : H^2(\Omega) \longrightarrow \tilde{V}_k^*$  preserves quadratic polynomials, it follows from (5.2.10) that

$$\begin{aligned} \|\tilde{\zeta} - \tilde{E}_k^* \tilde{\Pi}_k^* \tilde{\zeta}\|_{L_2(\tilde{T})} &= \|(\tilde{\zeta} - p) - \tilde{E}_k^* \tilde{\Pi}_k^*(\tilde{\zeta} - p)\|_{L_2(\tilde{T})} \\ &\lesssim \|\tilde{\zeta} - p\|_{H^2(S(\tilde{T}))} \lesssim \|\tilde{\zeta} - p\|_{H^{2+\alpha}(S(\tilde{T}))} \end{aligned}$$

for all  $\zeta \in H^{2+\alpha}(\Omega)$  and  $p \in P_2$ . The Bramble-Hilbert Lemma implies that

$$\|\tilde{\zeta} - \tilde{E}_k^* \tilde{\Pi}_k^* \tilde{\zeta}\|_{L_2(\tilde{T})} \lesssim |\tilde{\zeta}|_{H^{2+\alpha}(S(\tilde{T}))} \quad \forall \zeta \in H^{2+\alpha}(\Omega). \quad (5.2.11)$$

Similarly we have

$$|\tilde{\zeta} - \tilde{E}_k^* \tilde{\Pi}_k^* \tilde{\zeta}|_{H^2(\tilde{T})} \lesssim |\tilde{\zeta}|_{H^{2+\alpha}(S(\tilde{T}))} \quad \forall \zeta \in H^{2+\alpha}(\Omega). \quad (5.2.12)$$

From (5.2.11), (5.2.12) and a scaling argument we have

$$\|\zeta - E_k^* \Pi_k^* \zeta\|_{L_2(\Omega)} + h_k^2 |\zeta - E_k^* \Pi_k^* \zeta|_{H^2(\Omega)} \lesssim h_h^{2+\alpha} |\zeta|_{H^{2+\alpha}(\Omega)} \quad \forall \zeta \in H^{2+\alpha}(\Omega). \quad (5.2.13)$$

Let  $\zeta \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega)$ . Then  $E_k^* \Pi_k^* \zeta = E_k^* \Pi_k \zeta$ . Since  $E_k^* \Pi_k \zeta$  and  $E_k \Pi_k$  differ only by their first order derivatives at the vertices along  $\partial\Omega$ , we have

$$\begin{aligned} \|E_k^* \Pi_k \zeta - E_k \Pi_k \zeta\|_{L_2(\Omega)}^2 &= \sum_{T \in \mathcal{T}_k: \partial T \cap \partial\Omega \neq \emptyset} \|E_k^* \Pi_k \zeta - E_k \Pi_k \zeta\|_{L_2(T)}^2 \\ &\approx h_k^4 \sum_{p \in \partial\Omega} [\nabla(E_k^* \Pi_k \zeta - E_k \Pi_k \zeta)(p)]^2 \\ &= h_k^4 \sum_{p \in \partial\Omega} [\nabla(E_k^* \Pi_k \zeta)(p)]^2. \end{aligned} \quad (5.2.14)$$

From (5.2.13) we have

$$[\nabla(E_k^* \Pi_k \zeta)(p)]^2 = [\nabla(E_k^* \Pi_k^* \zeta)(p) - \nabla\zeta(p)]^2 \lesssim h_k^{2\alpha} \sum_{T \in S_p} |\zeta|_{H^{2+\alpha}(T)}^2, \quad (5.2.15)$$

where  $S_p$  is the set of triangles in  $\mathcal{T}_k$  sharing  $p$  as a vertex.

From (5.2.14) and (5.2.15) we have

$$\|E_k^* \Pi_k \zeta - E_k \Pi_k \zeta\|_{L_2(\Omega)}^2 \lesssim h_k^{4+2\alpha} |\zeta|_{H^{2+\alpha}(\Omega)}^2. \quad (5.2.16)$$

An inverse inequality implies that

$$\|E_k^* \Pi_k \zeta - E_k \Pi_k \zeta\|_{H^2(\Omega)}^2 \lesssim h_k^{2\alpha} |\zeta|_{H^{2+\alpha}(\Omega)}^2. \quad (5.2.17)$$

The lemma follows from (5.2.13), (5.2.16), (5.2.17) and triangle inequalities.  $\square$

By a similar argument we can establish the follow estimate.

$$\|\zeta - E_k \Pi_k \zeta\|_{L_2(\Omega)} \lesssim h_k^3 |\zeta|_{H^3(\Omega)} \quad \forall \zeta \in H^3(\Omega) \cap H_0^2(\Omega) \quad (5.2.18)$$

**Lemma 5.4** *Let  $\zeta \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega)$  and  $\zeta_k \in V_k$  be related by (4.1.21). Then*

$$|a_k(\zeta - \zeta_k, v)| \lesssim h_k^\alpha |\zeta|_{H^{2+\alpha}(\Omega)} \|v\|_{a_k} \quad \forall v \in V_k. \quad (5.2.19)$$

*Proof.* Suppose  $\zeta \in H_0^2(\Omega)$ . From (4.1.21) and a duality argument we have

$$\|\zeta_k\|_{a_k} \lesssim |\zeta|_{H^2(\Omega)}.$$

Therefore by a Cauchy-Schwarz inequality we have

$$|a_k(\zeta - \zeta_k, v)| \leq \|\zeta - \zeta_k\|_{a_k} \|\zeta\|_{a_k} \lesssim |\zeta|_{H^2(\Omega)} \|v\|_{a_k} \quad \forall v \in V_k. \quad (5.2.20)$$

Suppose  $\zeta \in H_0^2(\Omega) \cap H^3(\Omega)$ . First we have

$$\begin{aligned} a_k(\zeta - \zeta_k, v) &= a_k(\zeta, v) - a_k(\zeta_k, v) \\ &= a_k(\zeta, v) - a_k(\zeta, E_k v). \end{aligned} \quad (5.2.21)$$

By Green's formula we have

$$\begin{aligned} a_k(\zeta, v) &= \sum_{T \in \mathcal{T}_k} \int_T \sum_{i,j=1}^2 \frac{\partial^2 \zeta}{\partial x_i \partial x_j} \cdot \frac{\partial^2 v}{\partial x_i \partial x_j} dx \\ &= - \sum_{T \in \mathcal{T}_k} \int_T \nabla(\Delta \zeta) \cdot \nabla v dx + \sum_e \int_e (G_1(\zeta)[[v_{x_1}]] + G_2(\zeta)[[v_{x_2}]]) ds, \end{aligned} \quad (5.2.22)$$

where the second summation is taken over all the edges  $e$  of  $\mathcal{T}_k$ ,  $[[v_{x_1}]]$  and  $[[v_{x_2}]]$  denote the jumps of  $\partial v/\partial x_1$  and  $\partial v/\partial x_2$  crossing the edge  $e$ , and

$$\begin{aligned} G_1(\zeta) &= \frac{\partial^2 \zeta}{\partial x_1 \partial x_1} n_{e,1} + \frac{\partial^2 \zeta}{\partial x_1 \partial x_2} n_{e,2}, \\ G_2(\zeta) &= \frac{\partial^2 \zeta}{\partial x_2 \partial x_1} n_{e,1} + \frac{\partial^2 \zeta}{\partial x_2 \partial x_2} n_{e,2}, \end{aligned}$$

where  $(n_{e,1}, n_{e,2})$  is the unit vector perpendicular to the edge  $e$ .

Since  $E_k v \in H_0^2(\Omega)$ , it follows from (5.2.22) that

$$a_k(\zeta, E_k v) = - \sum_{T \in \mathcal{T}_k} \int_T \nabla(\Delta \zeta) \cdot \nabla E_k v \, dx. \quad (5.2.23)$$

We claim that

$$\left| \sum_e \int_e (G_1(\zeta)[[v_{x_1}]] + G_2(\zeta)[[v_{x_2}]]) \, ds \right| \lesssim h_k |\zeta|_{H^3(\Omega)} \|v\|_{a_k} \quad \forall v \in V_k. \quad (5.2.24)$$

From (4.1.7), (5.2.21)–(5.2.24), Lemma 5.2, an inverse estimate and a Cauchy-Schwarz inequality we have

$$\begin{aligned} a_k(\zeta - \zeta_k, v) &= \sum_{T \in \mathcal{T}_k} \int_T \nabla(\Delta \zeta) \cdot \nabla(v - E_k v) \, dx \\ &\quad + \sum_e \int_e (G_1(\zeta)[[v_{x_1}]] + G_2(\zeta)[[v_{x_2}]]) \, ds \\ &\lesssim h_k |\zeta|_{H^3(\Omega)} \|v - E_k v\|_{a_k} + h_k^{-1} |\zeta|_{H^3(\Omega)} \|v\|_{L_2(\Omega)} \\ &\lesssim h_k |\zeta|_{H^3(\Omega)} \|v\|_{a_k}. \end{aligned} \quad (5.2.25)$$

The estimate (5.2.19) follows from (5.2.20), (5.2.25) and interpolation.

We also need to verify the claim (5.2.24).

Let  $T_1$  and  $T_2$  be two triangles in  $\mathcal{T}_k$  with common edge  $e$ . Using  $\tilde{x} = x/h_k$ , we transform the triangles into reference domain (cf. Figure 5.4).

Let  $\tilde{\zeta}$  and  $\tilde{v}$  be the rescaling of  $\zeta$  and  $v$  on the reference domain. Let  $p \in P_2$  be a quadratic polynomial. Then  $G(p)$  is a constant. But

$$[[\tilde{v}_{x_1}]] = \frac{\partial \tilde{v}_{T_1}}{\partial x_1} \Big|_{\tilde{e}} - \frac{\partial \tilde{v}_{T_2}}{\partial x_1} \Big|_{\tilde{e}}$$

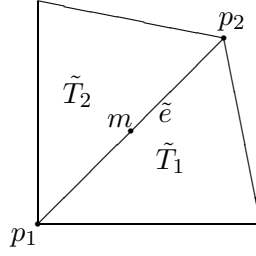


Figure 5.4: Two neighboring reference triangles  $\tilde{T}_1$  and  $\tilde{T}_2$  for  $\mathcal{T}_k$

is linear and is 0 at the midpoint  $m$  of  $\tilde{e}$ , Then

$$\int_{\tilde{e}} G_1(p)[[\tilde{v}_{x_1}]] ds = 0.$$

Therefore

$$\int_{\tilde{e}} G_1(\tilde{\zeta})[[\tilde{v}_{x_1}]] ds = \int_{\tilde{e}} G_1(\tilde{\zeta} - p)[[\tilde{v}_{x_1}]] ds$$

for all  $p \in P_2$ . By a Cauchy-Schwarz inequality we have

$$\left| \int_{\tilde{e}} G_1(\tilde{\zeta})[[\tilde{v}_{x_1}]] ds \right| \leq \inf_{p \in P_2} \|G_1(\tilde{\zeta} - p)\|_{L_2(\tilde{e})} \|[[\tilde{v}_{x_1}]]\|_{L_2(\tilde{e})}. \quad (5.2.26)$$

By a trace theorem (cf. [44]) and the Bramble-Hilbert Lemma we have

$$\inf_{p \in P_2} \|G_1(\tilde{\zeta} - p)\|_{L_2(\tilde{e})} \lesssim \inf_{p \in P_2} \|G_1(\tilde{\zeta} - p)\|_{H^2(\tilde{T}_1)} \lesssim |\tilde{\zeta}|_{H^3(\tilde{T}_1)}. \quad (5.2.27)$$

On the other hand, since  $\nabla \tilde{v}_{T_1}(m) = \nabla \tilde{v}_{T_2}(m)$ , we have

$$\begin{aligned} \|[[\tilde{v}_{x_1}]]\|_{L_2(\tilde{e})}^2 &= \left\| \frac{\partial \tilde{v}_{T_1}}{\partial x_1} - \frac{\partial \tilde{v}_{T_2}}{\partial x_1} \right\|_{L_2(\tilde{e})}^2 \\ &\approx \left[ \frac{\partial \tilde{v}_{T_1}}{\partial x_1}(p_1) - \frac{\partial \tilde{v}_{T_2}}{\partial x_1}(p_1) \right]^2 + \left[ \frac{\partial \tilde{v}_{T_1}}{\partial x_1}(p_2) - \frac{\partial \tilde{v}_{T_2}}{\partial x_1}(p_2) \right]^2 \\ &\lesssim \left[ \frac{\partial \tilde{v}_{T_1}}{\partial x_1}(p_1) - \frac{\partial \tilde{v}}{\partial x_1}(m) \right]^2 + \left[ \frac{\partial \tilde{v}}{\partial x_1}(m) - \frac{\partial \tilde{v}_{T_2}}{\partial x_1}(p_1) \right]^2 \\ &\quad + \left[ \frac{\partial \tilde{v}_{T_1}}{\partial x_1}(p_2) - \frac{\partial \tilde{v}}{\partial x_1}(m) \right]^2 + \left[ \frac{\partial \tilde{v}}{\partial x_1}(m) - \frac{\partial \tilde{v}_{T_2}}{\partial x_1}(p_2) \right]^2 \\ &\lesssim |\tilde{v}|_{H^2(\tilde{T}_1)}^2 + |\tilde{v}|_{H^2(\tilde{T}_2)}^2. \end{aligned} \quad (5.2.28)$$

From (5.2.26)–(5.2.28) we have

$$\left| \int_{\tilde{e}} G_1(\tilde{\zeta})[[\tilde{v}_{x_1}]] ds \right| \lesssim |\tilde{\zeta}|_{H^3(\tilde{T}_1)} \left( |\tilde{v}|_{H^2(\tilde{T}_1)} + |\tilde{v}|_{H^2(\tilde{T}_2)} \right).$$

A scaling argument yields

$$\left| \int_e G_1(\zeta)[[v_{x_1}]] ds \right| \lesssim h_k |\zeta|_{H^3(T_1)} (|v|_{H^2(T_1)} + |v|_{H^2(T_2)}).$$

In the case where  $e$  is a boundary edge of a triangle  $T$ , we can obtain in the same way that

$$\left| \int_e G_1(\zeta)[[v_{x_1}]] ds \right| \lesssim h_k |\zeta|_{H^3(T_1)} |v|_{H^2(T)}.$$

Summing up over all edges  $e$ 's in  $\mathcal{T}_k$  and using a Cauchy-Schwarz inequality give

$$\left| \sum_e \int_e (G_1(\zeta)[[v_{x_1}]] ds \right| \lesssim h_k |\zeta|_{H^3(\Omega)} \|v\|_{a_k} \quad \forall v \in V_k.$$

A similar estimate holds for  $|\sum_e \int_e (G_2(\zeta)[[v_{x_2}]] ds|$ . The claim (5.2.24) then follows.  $\square$

**Lemma 5.5** *Let  $\zeta \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega)$  and  $\zeta_k \in V_k$  be related by (4.1.21). Then*

$$|a_k(\zeta - \zeta_k, \Pi_k \xi)| \lesssim h_k^{2\alpha} |\zeta|_{H^{2+\alpha}(\Omega)} |\xi|_{H^{2+\alpha}(\Omega)} \quad \forall \xi \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega). \quad (5.2.29)$$

*Proof.* First we have by a Cauchy-Schwarz inequality, (5.1.9) and (5.2.20) that

$$|a_k(\zeta - \zeta_k, \Pi_k \xi)| \lesssim |\zeta|_{H^2(\Omega)} |\xi|_{H^2(\Omega)} \quad \forall \zeta, \xi \in H_0^2(\Omega). \quad (5.2.30)$$

Secondly, let  $\zeta, \xi \in H_0^2(\Omega) \cap H^3(\Omega)$ . From (5.2.21) and Green's formula we have

$$\begin{aligned} a_k(\zeta - \zeta_k, \Pi_k \xi) &= a_k(\zeta, \Pi_k \xi - E_k \Pi_k \xi) \\ &= \sum_{T \in \mathcal{T}_k} \int_T \sum_{i,j=1}^2 \frac{\partial^2 \zeta}{\partial x_i \partial x_j} \cdot \frac{\partial^2 (\Pi_k \xi - E_k \Pi_k \xi)}{\partial x_i \partial x_j} dx \\ &= - \sum_{T \in \mathcal{T}_k} \int_T \nabla(\Delta \zeta) \cdot \nabla(\Pi_k \xi - E_k \Pi_k \xi) dx \\ &\quad + \sum_e \int_e (G_1(\zeta)[[(\Pi_k \xi - E_k \Pi_k \xi)_{x_1}]] + G_2(\zeta)[[(\Pi_k \xi - E_k \Pi_k \xi)_{x_2}]] ds. \end{aligned} \quad (5.2.31)$$

By approximation theory we have

$$\|\Pi_k v - v\|_{L_2(\Omega)} \lesssim |v|_{H^3(\Omega)} \quad \forall v \in H^3(\Omega) \cap H_0^2(\Omega). \quad (5.2.32)$$

From (5.2.18), (5.2.32) and a triangle inequality we have

$$\|\Pi_k \xi - E_k \Pi_k \xi\|_{L_2(\Omega)} \leq \|\Pi_k \xi - \xi\|_{L_2(\Omega)} + \|\xi - E_k \Pi_k \xi\|_{L_2(\Omega)} \lesssim h_k^3 \|\xi\|_{H^3(\Omega)}.$$

Therefor from an inverse estimate we have

$$\left( \sum_{T \in \mathcal{T}_k} |E_k \Pi_k \xi - \Pi_k \xi|_{H^1(T)}^2 \right)^{1/2} \lesssim h_k^2 \|\xi\|_{H^3(\Omega)}. \quad (5.2.33)$$

Then by (5.2.33) and a Cauchy-Schwarz inequality we have

$$\left| \sum_{T \in \mathcal{T}_k} \int_T \nabla(\Delta \zeta) \cdot \nabla(\Pi_k \xi - E_k \Pi_k \xi) dx \right| \lesssim h_k^2 |\zeta|_{H^3(\Omega)} |\xi|_{H^3(\Omega)}. \quad (5.2.34)$$

By using an argument similar to the proof of (5.2.24) we have

$$\begin{aligned} & \left| \sum_e \int_e (G_1(\zeta)[[(\Pi_k \xi - E_k \Pi_k \xi)_{x_1}]] + G_2(\zeta)[[(\Pi_k \xi - E_k \Pi_k \xi)_{x_2}]] ds \right| \\ & \lesssim h_k |\zeta|_{H^3(\Omega)} \|\Pi_k \xi - E_k \Pi_k \xi\|_{a_k} \lesssim h_k^2 |\zeta|_{H^3(\Omega)} |\xi|_{H^3(\Omega)}. \end{aligned} \quad (5.2.35)$$

From (5.2.31), (5.2.34) and (5.2.35) we have

$$a_k(\zeta - \zeta_k, \Pi_k \xi) \lesssim h_k^2 |\zeta|_{H^3(\Omega)} |\xi|_{H^3(\Omega)}. \quad (5.2.36)$$

The estimate (5.2.29) follows from (5.2.30), (5.2.36) and a bilinear interpolation.  $\square$

The estimates (5.2.19) and (5.2.29) establish Assumption 5.

### 5.3 The Morley Interpolation Operators

We have known that the Morley interpolation operators satisfies (4.1.16) and (4.1.17).

In this section we discuss some further properties of the operators. We first extend the operator  $\Pi_k : H_0^2(\Omega) \longrightarrow V_k$  to a larger space  $H_0^2(\Omega) + V_{k-1} + V_k$ .

Let  $v \in H_0^2(\Omega) + V_k + V_{k+1}$ . First of all, the value  $v(p)$  is well-defined for  $p \in \mathcal{V}_k$ . Secondly, the integral  $\int_e \frac{\partial v}{\partial n} ds$  is also well-defined for  $e \in \mathcal{E}_k$ . In particular, if  $v \in V_{k+1}$ , then

$$\int_e \frac{\partial v}{\partial n} ds = \int_{e_1} \frac{\partial v}{\partial n} ds + \int_{e_2} \frac{\partial v}{\partial n} ds \quad (5.3.1)$$

where  $e_1, e_2 \in \mathcal{E}_{k+1}$  with  $e = e_1 \cup e_2$  (cf. Figure 5.5). Therefore the linear operator  $\Pi_k$  is well-defined from the larger space  $H_0^2(\Omega) + V_k + V_{k+1}$  into  $V_k$ . In particular,  $\Pi_k : H_0^2(\Omega) + V_k \rightarrow V_k$  and  $\Pi_{k-1} : H_0^2(\Omega) + V_k \rightarrow V_{k-1}$  are both well-defined.

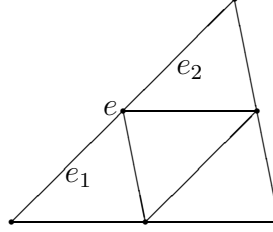


Figure 5.5: An edge  $e \in \mathcal{E}_{k-1}$ .

**Lemma 5.6** *It holds that*

$$\|v - \Pi_k v\|_{L_2(\Omega)} + h_k^2 \|\Pi_k v\|_{a_k} \lesssim h_k^2 \|v\|_{a_k} \quad \forall v \in H_0^2(\Omega) + V_k. \quad (5.3.2)$$

*Proof.* Let  $T \in \mathcal{T}_k$ ,  $\zeta \in H^2(T)$  and the quadratic polynomial  $\Pi_T \zeta$  on  $T$  be the Morley nodal interpolant of  $\zeta$ , i.e.,

$$(\Pi_T \zeta)(p_i) = \zeta(p_i) \text{ and } \int_{e_i} \frac{\partial(\Pi_T \zeta)}{\partial n} ds = \int_{e_i} \frac{\partial \zeta}{\partial n} ds, \quad (5.3.3)$$

for  $i = 1, 2$  and  $3$ , where  $p_1, p_2$  and  $p_3$  are the vertices of  $T$ , and  $e_1, e_2$  and  $e_3$  are the edges of  $T$ . It is well known that (cf. [27] and [31]),

$$\|\zeta - \Pi_T \zeta\|_{L_2(T)} + h_k^2 |\Pi_T \zeta|_{H^2(T)} \lesssim h_k^2 |\zeta|_{H^2(T)}. \quad (5.3.4)$$

Let  $v \in H_0^2(\Omega) + V_k$  and  $T \in \mathcal{T}_k$ . Then  $v_T \in H^2(T)$  and  $\Pi_k v = \Pi_T v_T$  on  $T$ . Therefore

$$\|v - \Pi_k v\|_{L_2(T)} + h_k^2 |\Pi_k v|_{H^2(T)} \lesssim h_k^2 |v|_{H^2(T)}. \quad (5.3.5)$$

The estimate (5.3.2) holds because (5.3.5) is valid for all  $T \in \mathcal{T}_k$ .  $\square$

**Lemma 5.7** *It holds that*

$$\Pi_{k-1} \Pi_k v = \Pi_{k-1} v \quad \forall v \in H_0^2(\Omega) + V_k. \quad (5.3.6)$$

*Proof.* Let  $v \in H_0^2(\Omega) + V_k$  be arbitrary. The functions  $\Pi_{k-1}\Pi_k v$  and  $\Pi_{k-1}v$  are both in  $V_{k-1}$ . Moreover, we have

$$(\Pi_{k-1}\Pi_k v)(p) = (\Pi_{k-1}v)(p)$$

for all  $p \in \mathcal{V}_{k-1}$ , and

$$\begin{aligned} \int_e \frac{\partial(\Pi_{k-1}\Pi_k v)}{\partial n} ds &= \int_{e_1} \frac{\partial(\Pi_k v)}{\partial n} ds + \int_{e_2} \frac{\partial(\Pi_k v)}{\partial n} ds \\ &= \int_{e_1} \frac{\partial v}{\partial n} ds + \int_{e_2} \frac{\partial v}{\partial n} ds \\ &= \int_e \frac{\partial v}{\partial n} ds = \int_e \frac{\partial(\Pi_{k-1}v)}{\partial n} ds \end{aligned}$$

for all  $e \in \mathcal{E}_{k-1}$ , where  $e_1, e_2 \in \mathcal{E}_k$  with  $e = e_1 \cup e_2$  (cf. Figure 5.5). Therefore  $\Pi_{k-1}\Pi_k v = \Pi_{k-1}v$ .  $\square$

**Lemma 5.8** *Assumption 6 of Chapter 4 holds. That is*

$$\|\Pi_{k-1}v - v\|_{L_2(\Omega)} \lesssim h_k^2 \|v\|_{a_k} \quad \forall v \in V_k.$$

*Proof.* Let  $T \in \mathcal{T}_{k-1}$  be divided into 4 triangles  $T_1, T_2, T_3$  and  $T_4$  in  $\mathcal{T}_k$  and  $\tilde{T} = T/h_{k-1}$ . Then  $|\tilde{T}| \approx 1$  (cf. Figure 5.6).

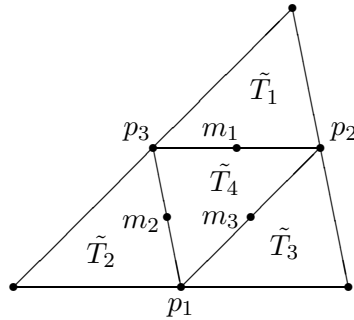


Figure 5.6: A reference triangle  $\tilde{T}$  divided into 4 triangles  $\tilde{T}_1, \tilde{T}_2, \tilde{T}_3$  and  $\tilde{T}_4$

For each  $v \in V_k$ , define  $\tilde{v}(\tilde{x}) = v(h_{k-1}\tilde{x})$  for  $\tilde{x} \in \tilde{T}$ . Note that  $\tilde{x} \in \tilde{T}$  if and only if  $h_{k-1}\tilde{x} \in T$ . If  $w = \Pi_{k-1}v$ , then we define  $\tilde{\Pi}_{k-1}\tilde{v}$  to be  $\tilde{w}$ .

Let  $V(\tilde{T})$  be the Morley finite element space associated with  $\tilde{T}_1, \tilde{T}_2, \tilde{T}_3$  and  $\tilde{T}_4$ . Note that  $V(\tilde{T})$  is the space of functions  $\tilde{v} \in L_2(\tilde{T})$  such that  $\tilde{v}|_{\tilde{T}_i}$  is a quadratic polynomial on  $\tilde{T}_i$  for  $i = 1, 2, 3$  and  $4$ ,  $\tilde{v}$  is continuous at  $p_1, p_2$  and  $p_3$ , and  $\partial\tilde{v}/\partial n$  is continuous at  $m_1, m_2$  and  $m_3$ . We can see that  $V(\tilde{T})$  is a finite dimensional linear space and

$$\|\tilde{v}\|_* = \left[ \sum_{i=1}^4 |\tilde{v}|_{H^2(\tilde{T}_i)}^2 \right]^{1/2}$$

defines a norm on the quotient space  $V(\tilde{T})/P_1(\tilde{T})$ , where  $P_1(\tilde{T})$  is the space of linear functions on  $\tilde{T}$ . On the other hand,

$$v \longrightarrow \|\tilde{\Pi}_{k-1}\tilde{v} - \tilde{v}\|_{L_2(\tilde{T})}$$

defines a semi-norm on  $V(\tilde{T})/P_1(\tilde{T})$ . Therefore

$$\|\tilde{\Pi}_{k-1}\tilde{v} - \tilde{v}\|_{L_2(\tilde{T})} \lesssim \left[ \sum_{i=1}^4 |\tilde{v}|_{H^2(\tilde{T}_i)}^2 \right]^{1/2}. \quad (5.3.7)$$

A scaling argument on (5.3.7) yields

$$\|\Pi_{k-1}v - v\|_{L_2(T)} \lesssim h_k^2 \left[ \sum_{i=1}^4 |v|_{H^2(T_i)}^2 \right]^{1/2}. \quad (5.3.8)$$

The lemma follows. □

**Lemma 5.9** *Assumption 7 of Chapter 4 holds. That is,*

$$\|\Pi_{k-1}v\|_{a_k} \lesssim \|v\|_{a_k} \quad \forall v \in H_0^2(\Omega) + V_k.$$

*Proof.* From (5.3.2), (5.3.6), Assumption 6 and an inverse estimate we have that,

for all  $v \in H_0^2(\Omega) + V_k$ ,

$$\begin{aligned}
\|\Pi_{k-1}v\|_{a_k} &= \|\Pi_{k-1}\Pi_k v\|_{a_k} \\
&\leq \|\Pi_{k-1}\Pi_k v - \Pi_k v\|_{a_k} + \|\Pi_k v\|_{a_k} \\
&\lesssim h_k^{-2} \|\Pi_{k-1}\Pi_k v - \Pi_k v\|_{L_2(\Omega)} + \|\Pi_k v\|_{a_k} \\
&\lesssim \|\Pi_k v\|_{a_k} \lesssim \|v\|_{a_k}.
\end{aligned}$$

□

**Lemma 5.10** *Assumption 8 of Chapter 4 holds. That is,*

$$\|\|\Pi_{k-1}v\|_{0,k-1}^2 \leq (1 + \theta^2) \|\|v\|_{0,k}^2 + C\theta^{-2}h_k^{2\alpha} \|\|v\|_{\alpha,k}^2 \quad (5.3.9)$$

for all  $v \in V_k$  and  $\theta \in (0, 1)$ .

*Proof.* Let  $v \in V_k$  be arbitrary. It is easy to see from (5.1.3) that  $\|\|v\|_{0,k}^2$  can be expressed as follows:

$$\|\|v\|_{0,k}^2 = h_k^2 \left[ \frac{1}{6} \sum_{T \in \mathcal{T}_k} \sum_{p \in \mathcal{V}_T} v(p)^2 + \sum_{e \in \mathcal{E}_k} \left( \int_e \frac{\partial v}{\partial n} ds \right)^2 \right], \quad (5.3.10)$$

where  $\mathcal{V}_T$  is the set of the vertices of the triangle  $T$ .

Let  $w = \Pi_{k-1}v$ . Then

$$\|\|w\|_{0,k-1}^2 = h_{k-1}^2 \left[ \frac{1}{6} \sum_{T \in \mathcal{T}_{k-1}} \sum_{p \in \mathcal{V}_T} w(p)^2 + \sum_{e \in \mathcal{E}_{k-1}} \left( \int_e \frac{\partial w}{\partial n} ds \right)^2 \right]. \quad (5.3.11)$$

By the definition (5.1.8) of  $\Pi_{k-1}$  and (5.3.1), we have  $w(p) = v(p)$  for all  $p \in \mathcal{V}_{k-1}$  and

$$\begin{aligned}
\left( \int_e \frac{\partial w}{\partial n} ds \right)^2 &= \left( \int_{e_1} \frac{\partial v}{\partial n} ds + \int_{e_2} \frac{\partial v}{\partial n} ds \right)^2 \\
&\leq 2 \left( \int_{e_1} \frac{\partial v}{\partial n} ds \right)^2 + 2 \left( \int_{e_2} \frac{\partial v}{\partial n} ds \right)^2
\end{aligned}$$

for all  $e \in \mathcal{E}_{k-1}$ , where  $e_1, e_2 \in \mathcal{E}_k$  with  $e = e_1 \cup e_2$  (cf. Figure 5.5). Therefore

$$\|\Pi_{k-1}v\|_{0,k-1}^2 \leq \frac{h_{k-1}^2}{6} \sum_{T \in \mathcal{T}_{k-1}} \sum_{p \in \mathcal{V}_T} v(p)^2 + Ch_k^2 \sum_{e \in \mathcal{E}_k} \left( \int_e \frac{\partial v}{\partial n} ds \right)^2. \quad (5.3.12)$$

Let  $T \in \mathcal{T}_{k-1}$  be divided into four triangles  $T_1, T_2, T_3$  and  $T_4$  in  $\mathcal{T}_k$ , whose vertices are labeled as in Figure 5.7. Then we have

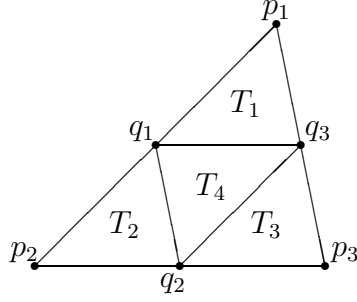


Figure 5.7: A Triangle  $T \in \mathcal{T}_{k-1}$  divided into four triangles in  $\mathcal{T}_k$

$$\begin{aligned} 4 \sum_{p \in \mathcal{V}_T} v(p)^2 &= \sum_{i=1}^3 v(p_i)^2 + 3 \sum_{i=1}^3 [v(q_i) + (v(p_i) - v(q_i))]^2 \\ &\leq \sum_{i=1}^3 v(p_i)^2 + 3 \sum_{i=1}^3 [(1 + \theta^2)v(q_i)^2 + (1 + \theta^{-2})(v(p_i) - v(q_i))^2] \quad (5.3.13) \\ &\leq (1 + \theta^2) \sum_{i=1}^4 \sum_{p \in \mathcal{V}_{T_i}} v(p)^2 + C\theta^{-2} \sum_{i=1}^4 |v|_{H^1(T_i)}^2. \end{aligned}$$

From (1.2.1) and (5.3.13) we have

$$h_{k-1}^2 \sum_{p \in \mathcal{V}_T} v(p)^2 \leq (1 + \theta^2)h_k^2 \sum_{i=1}^4 \sum_{p \in \mathcal{V}_{T_i}} v(p)^2 + C\theta^{-2}h_k^2 \sum_{i=1}^4 |v|_{H^1(T_i)}^2. \quad (5.3.14)$$

Summing up over all  $T \in \mathcal{T}_{k-1}$  gives

$$\begin{aligned} h_{k-1}^2 \sum_{T \in \mathcal{T}_{k-1}} \sum_{p \in \mathcal{V}_T} v(p)^2 & \quad (5.3.15) \\ &\leq h_k^2(1 + \theta^2) \sum_{T \in \mathcal{T}_k} \sum_{p \in \mathcal{V}_T} v(p)^2 + C\theta^{-2}h_k^2 \sum_{T \in \mathcal{T}_k} |v|_{H^1(T)}^2. \end{aligned}$$

Using a similar argument as we did for (3.3.7), we have

$$\sum_{e \in \mathcal{E}_k} \left( \int_e \frac{\partial v}{\partial n} ds \right)^2 \leq C \sum_{T \in \mathcal{T}_k} |v|_{H^1(T)}^2. \quad (5.3.16)$$

Therefore from (2.1.32), Lemma 4.3, (5.3.10), (5.3.12), (5.3.15) and (5.3.16) we have

$$\begin{aligned} \|\Pi_{k-1} v\|_{0,k-1}^2 &\leq \frac{h_{k-1}^2}{6} \sum_{T \in \mathcal{T}_{k-1}} \sum_{p \in \mathcal{V}_T} v(p)^2 + Ch_k^2 \sum_{e \in \mathcal{E}_k} \left( \int_e \frac{\partial v}{\partial n} ds \right)^2 \\ &\leq \frac{h_k^2}{6} (1 + \theta^2) \sum_{T \in \mathcal{T}_k} \sum_{p \in \mathcal{V}_T} v(p)^2 + C\theta^{-2} h_k^2 \sum_{T \in \mathcal{T}_k} |v|_{H^1(T)}^2 \\ &\leq (1 + \theta^2) \|v\|_{0,k}^2 + C\theta^{-2} h_k^2 \|v\|_{1,k}^2 \\ &\leq (1 + \theta^2) \|v\|_{0,k}^2 + C\theta^{-2} h_k^{2\alpha} \|v\|_{\alpha,k}^2. \end{aligned}$$

□

## 5.4 Intergrid Transfer Operators

In this section we prove the last three Assumptions in Chapter 4 about the intergrid transfer operators.

**Lemma 5.11** *Assumption 9 of Chapter 4 holds. That is,*

$$\|I_{k-1}^k v - v\|_{L_2(\Omega)} \lesssim h_k^2 |v|_{a_{k-1}} \quad \forall v \in V_{k-1}. \quad (5.4.1)$$

*Proof.* Let  $v \in V_k$  be arbitrary and  $T$  a strangle in  $\mathcal{T}_k$ . Let  $T_1$  and  $T_2$  be two triangles in  $\mathcal{T}_{k-1}$  such that  $T \subset T_1$  and  $T_2$  neighbors  $T_1$  (cf. Figure 5.8).

Since  $I_{k-1}^k v - v$  is a quadratic polynomial on  $T$ , it follows that

$$\begin{aligned} \|I_{k-1}^k v - v\|_{L_2(T)}^2 &\approx h_k^2 \sum_{j=1}^3 [I_{k-1}^k v(p_j) - v_T(p_j)]^2 \\ &\quad + h_k^4 \sum_{j=1}^3 \left[ \frac{\partial I_{k-1}^k v}{\partial n}(m_j) - \frac{\partial v_T}{\partial n}(m_j) \right]^2. \end{aligned} \quad (5.4.2)$$

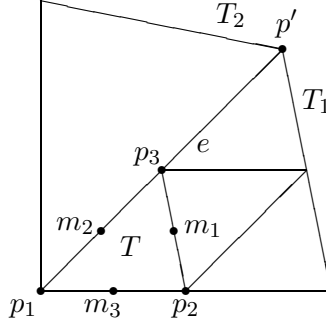


Figure 5.8: Two neighboring triangles  $T_1$  and  $T_2$  in  $\mathcal{T}_{k-1}$

Since  $p_1$  is a node in  $\mathcal{T}_{k-1}$ , we have  $I_{k-1}^k v(p_1) - v(p_1) = 0$ .

We consider the the jump  $w = v_{T_1} - v_{T_2}$  on the edge  $e$ . Since  $w$  is a quadratic polynomial on  $e$  with  $w(p_1) = w(p') = 0$ , it is easy to see from the Mean-Value Theorem that

$$|w(x)| \leq |e|^2 \sup_{t \in e} |w''(t)| \lesssim h_k (|v_{T_1}|_{H^2(T_1)} + |v_{T_2}|_{H^2(T_2)})$$

for all  $x$  on  $e$ . Therefore

$$\begin{aligned} |I_{k-1}^k v(p_3) - v_T(p_3)| &= \left| \frac{1}{2} [v_{T_1}(p_3) + v_{T_2}(p_3)] - v_{T_1}(p_3) \right| \\ &= \left| \frac{1}{2} w(p_3) \right| \lesssim h_k (|v_{T_1}|_{H^2(T_1)} + |v_{T_2}|_{H^2(T_2)}). \end{aligned}$$

A similar estimate holds for  $|I_{k-1}^k v(p_2) - v_T(p_2)|$ .

As for normal derivatives at  $m_j$ , we first have

$$\frac{\partial I_{k-1}^k v}{\partial n}(m_1) - \frac{\partial v_T}{\partial n}(m_1) = 0.$$

Since the jump of normal derivatives  $z = \partial I_{k-1}^k v / \partial n - \partial v_T / \partial n$  on  $e$  vanishes at the midpoint  $p_3$  of  $e$ , we have

$$|z(x)| \leq |e| \sup_{t \in e} |z'(x)| \lesssim |v_{T_1}|_{H^2(T_1)} + |v_{T_2}|_{H^2(T_2)}$$

for all  $x$  on  $e$ . Therefore

$$\begin{aligned} \left| \frac{\partial I_{k-1}^k v}{\partial n}(m_2) - \frac{\partial v_T}{\partial n}(m_2) \right| &= \left| \frac{1}{2} \left[ \frac{\partial v_{T_1}}{\partial n} + \frac{\partial v_{T_2}}{\partial n} \right] - \frac{\partial v_{T_1}}{\partial n}(m_2) \right| \\ &= \left| \frac{1}{2} z(m_2) \right| \lesssim |v_{T_1}|_{H^2(T_1)} + |v_{T_2}|_{H^2(T_2)}. \end{aligned}$$

A similar estimate holds for the jump of the normal derivatives at  $m_3$ .

Putting everything together we have

$$\|I_{k-1}^k v - v\|_{L_2(T)}^2 \lesssim h_k^4 \sum_{T' \in S(T_1)} |v|_{H^2(T')}^2,$$

where  $S(T_1)$  is the triangle  $T_1$  and other neighboring triangles in  $\mathcal{T}_{k-1}$ . Summing up over all  $T \in \mathcal{T}_k$  proves the lemma.  $\square$

**Lemma 5.12** *Assumption 10 of Chapter 4 holds. That is,*

$$\|I_{k-1}^k \Pi_{k-1} \zeta - \Pi_k \zeta\|_{L_2(\Omega)} \lesssim h_k^{2+\alpha} |\zeta|_{H^{2+\alpha}(\Omega)} \quad \forall \zeta \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega). \quad (5.4.3)$$

*Proof.* Let  $\Pi_k^* : H^{2+\alpha}(\Omega) \rightarrow V_k^*$  be defined as in the proof of Assumption 4. We also define  $\mathcal{I}_{k-1}^k : V_{k-1}^* \rightarrow V_k^*$  to be the one similar to  $I_{k-1}^k$  but without boundary condition.

By using an argument similar to the proof for (5.2.13) we have

$$\|\mathcal{I}_{k-1}^k \Pi_{k-1}^* \zeta - \Pi_k^* \zeta\|_{L_2(\Omega)} \lesssim h_k^{2+\alpha} |\zeta|_{H^{2+\alpha}(\Omega)} \quad \forall \zeta \in H^{2+\alpha}(\Omega).$$

Let  $\zeta \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega)$ . Then  $\mathcal{I}_{k-1}^k \Pi_{k-1}^* \zeta - \Pi_k^* \zeta = \mathcal{I}_{k-1}^k \Pi_{k-1} \zeta - \Pi_k \zeta$ . Therefore

$$\begin{aligned} \|I_{k-1}^k \Pi_{k-1} \zeta - \Pi_k \zeta\|_{L_2(\Omega)} &\leq \|I_{k-1}^k \Pi_{k-1} \zeta - \mathcal{I}_{k-1}^k \Pi_{k-1} \zeta\|_{L_2(\Omega)} + \|\mathcal{I}_{k-1}^k \Pi_{k-1} \zeta - \Pi_k \zeta\|_{L_2(\Omega)} \\ &\lesssim \|I_{k-1}^k \Pi_{k-1} \zeta - \mathcal{I}_{k-1}^k \Pi_{k-1} \zeta\|_{L_2(\Omega)} + h_k^{2+\alpha} |\zeta|_{H^{2+\alpha}(\Omega)} \end{aligned}$$

for all  $\zeta \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega)$ .

Using an argument similar to (5.2.14)–(5.2.16), we have

$$\|I_{k-1}^k \Pi_{k-1} \zeta - \mathcal{I}_{k-1}^k \Pi_{k-1} \zeta\|_{L_2(\Omega)} \lesssim h_k^{2+\alpha} |\zeta|_{H^{2+\alpha}(\Omega)} \quad \forall \zeta \in H_0^2(\Omega) \cap H^{2+\alpha}(\Omega).$$

The lemma then follows.  $\square$

**Lemma 5.13** *Assumption 11 of Chapter 4 holds. That is,*

$$\|I_{k-1}^k v\|_{0,k}^2 \leq (1 + \theta^2) \|v\|_{0,k-1}^2 + C\theta^{-2} h_k^{2\alpha} \|v\|_{\alpha,k-1}^2 \quad (5.4.4)$$

for all  $v \in V_{k-1}$  and  $\theta \in (0, 1)$ .

*Proof.* Let  $v \in V_{k-1}$  be arbitrary and  $w = I_{k-1}^k v$ . Then by (5.1.3) we have

$$\|I_{k-1}^k v\|_{0,k}^2 = (w, w)_k = h_k^2 \left[ \sum_{p \in \mathcal{V}_k} n(p) w(p)^2 + \sum_{e \in \mathcal{E}_k} \left( \int_e \frac{\partial w}{\partial n} ds \right)^2 \right] \quad (5.4.5)$$

and

$$\|v\|_{0,k-1}^2 = h_k^2 \left[ \sum_{p \in \mathcal{V}_{k-1}} n(p) v(p)^2 + \sum_{e \in \mathcal{E}_{k-1}} \left( \int_e \frac{\partial v}{\partial n} ds \right)^2 \right], \quad (5.4.6)$$

where  $n(p) = |S_p|/6$  and  $S_p$  is the set of triangles sharing  $p$  as a common vertex. Note that  $n(p)$  is independent of  $k$ .

If  $p \in \mathcal{V}_{k-1}$ , then the value  $v(p)$  is well defined, i.e.,  $v_T(p) = v|_T(p) = v(p)$  for all  $T \in \mathcal{T}_{k-1}$  sharing  $p$  as a common vertex. From (5.1.6) in the definition of  $I_{k-1}^k$  we have  $w(p) = v(p)$ . If  $p \in \mathcal{V}_k \setminus \mathcal{V}_{k-1}$ . Then  $p$  is the midpoint of some  $e \in \mathcal{E}_{k-1}$ , which is the common edge of two triangles  $T, T' \in \mathcal{T}_{k-1}$  (cf. Figure 5.9). After subdivision,  $p$  is always the common vertex of 6 triangles in  $\mathcal{T}_k$  and therefore  $n(p) = 1$ . Hence we can write

$$\sum_{p \in \mathcal{V}_k} n(p) w(p)^2 = \sum_{p \in \mathcal{V}_{k-1}} n(p) v(p)^2 + \sum_{p \in \mathcal{V}_k \setminus \mathcal{V}_{k-1}} w(p)^2. \quad (5.4.7)$$

Suppose  $p_1$  and  $p_2$  are the endpoints of  $e$  (cf. Figure 5.9). We have

$$w(p)^2 = \left[ \frac{1}{2} (v_T(p) + v_{T'}(p)) \right]^2 \leq \frac{1}{2} (v_T(p))^2 + \frac{1}{2} (v_{T'}(p))^2. \quad (5.4.8)$$

Then from (2.2.2) we can write

$$\begin{aligned} \frac{1}{2} (v_T(p))^2 &= \frac{1}{2} [v(p_1) + (v_T(p) - v(p_1))]^2 \\ &\leq \frac{1}{2} (1 + \theta^2) v(p_1)^2 + C\theta^{-2} [v_T(p) - v(p_1)]^2. \end{aligned}$$

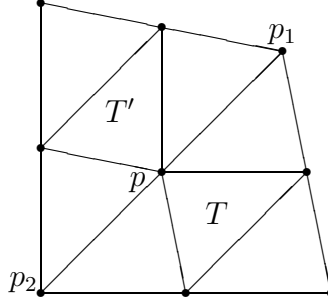


Figure 5.9: A vertex  $p \in \mathcal{T}_k \setminus \mathcal{T}_{k-1}$

Note that the Mean-Value Theorem and a standard inverse estimate imply that

$$[v_T(p) - v(p_1)]^2 \leq |p - p_1|^2 \|\nabla v\|_{L^\infty(T)}^2 \leq C|v|_{H^1(T)}^2.$$

Therefore we have

$$\frac{1}{2}(v_T(p))^2 \leq \frac{1}{2}(1 + \theta^2)v(p_1)^2 + C\theta^{-2}|v|_{H^1(T)}^2, \quad (5.4.9)$$

and similarly

$$\frac{1}{2}(v_{T'}(p))^2 \leq \frac{1}{2}(1 + \theta^2)v(p_2)^2 + C\theta^{-2}|v|_{H^1(T')}^2. \quad (5.4.10)$$

Thus from (5.4.8), (5.4.9), and (5.4.10) we have

$$w(p)^2 \leq \frac{1}{2}(1 + \theta^2)[v(p_1)^2 + v(p_2)^2] + C\theta^{-2}[|v|_{H^1(T)}^2 + |v|_{H^1(T')}^2]. \quad (5.4.11)$$

Taking summation of (5.4.11) over  $p \in \mathcal{V}_k \setminus \mathcal{V}_{k-1}$  gives

$$\begin{aligned} \sum_{p \in \mathcal{V}_k \setminus \mathcal{V}_{k-1}} w(p)^2 &\leq \frac{1}{2}(1 + \theta^2) \sum_{p \in \mathcal{V}_{k-1}} |S_p|v(p)^2 + C\theta^{-2} \sum_{T \in \mathcal{T}_{k-1}} |v|_{H^1(T)}^2 \\ &= 3(1 + \theta^2) \sum_{p \in \mathcal{V}_{k-1}} n(p)v(p)^2 + C\theta^{-2} \sum_{T \in \mathcal{T}_{k-1}} |v|_{H^1(T)}^2. \end{aligned}$$

Therefore it follows from (5.4.7) that

$$\sum_{p \in \mathcal{V}_k} n(p)w(p)^2 \leq 4(1 + \theta^2) \sum_{p \in \mathcal{V}_{k-1}} n(p)v(p)^2 + C\theta^{-2} \sum_{T \in \mathcal{T}_{k-1}} |v|_{H^1(T)}^2. \quad (5.4.12)$$

By the definition of  $I_{k-1}^k$  (cf. (5.1.7)), we have

$$\sum_{e \in \mathcal{E}_k} \left( \int_e \frac{\partial w}{\partial n} ds \right)^2 \leq C \sum_{T \in \mathcal{T}_k} \left( \int_{\partial T} \frac{\partial v_T}{\partial n} ds \right)^2. \quad (5.4.13)$$

From the Mean-Value Theorem and a standard inverse estimate we have

$$\left( \int_{\partial T} \frac{\partial v_T}{\partial n} ds \right)^2 \leq |\partial T|^2 \|\nabla v\|_{L^\infty(T)}^2 \leq C |v|_{H^1(T)}^2 \quad (5.4.14)$$

for all  $T \in \mathcal{T}_k$ . Therefore from (5.4.13) and (5.4.14) we have

$$\sum_{e \in \mathcal{E}_k} \left( \int_e \frac{\partial w}{\partial n} ds \right)^2 \leq C \sum_{T \in \mathcal{T}_{k-1}} |v|_{H^1(T)}^2. \quad (5.4.15)$$

By (1.2.1), (2.1.32), Corollary 4.3, (5.4.5), (5.4.6), (5.4.12) and (5.4.15), we have

$$\begin{aligned} & \|I_{k-1}^k v\|_{0,k}^2 \\ & \leq h_k^2 \left[ 4(1 + \theta^2) \sum_{p \in \mathcal{V}_{k-1}} n(p)v(p)^2 + C\theta^{-2} \sum_{T \in \mathcal{T}_{k-1}} |v|_{H^1(T)}^2 + C \sum_{T \in \mathcal{T}_{k-1}} |v|_{H^1(T)}^2 \right] \\ & \leq (1 + \theta^2)h_{k-1}^2 \sum_{p \in \mathcal{V}_{k-1}} n(p)v(p)^2 + C\theta^{-2}h_k^2 \sum_{T \in \mathcal{T}_{k-1}} |v|_{H^1(T)}^2 \\ & \leq (1 + \theta^2)\|v\|_{0,k-1}^2 + C\theta^{-2}h_k^2\|v\|_{1,k-1}^2 \\ & \leq (1 + \theta^2)\|v\|_{0,k-1}^2 + C\theta^{-2}h_k^{2\alpha}\|v\|_{\alpha,k-1}^2. \end{aligned}$$

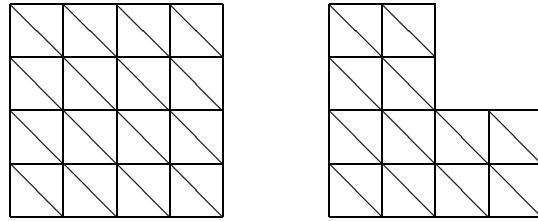
□

**Historical Remark** All of the estimates in Section 5.2 and (5.4.3) were established in [24]. The estimate (5.4.1) was obtained in [21]. The rest of the estimates in Sections 5.3 and 5.4 are new ones (cf. [54]).

## 5.5 Numerical Results

In this section we present the results of some numerical experiments to illustrate Theorems 2.13 and 2.14 for the Morley method.

First let  $\Omega$  be the unit square  $(0, 1) \times (0, 1)$  (cf. Figure 5.10(a)).



(a) Square domain      (b) L-shaped domain

Figure 5.10: The triangulation  $\mathcal{T}_k$  for  $k = 2$ .

Since the domain  $\Omega$  is convex, we have full elliptic regularity, i.e., the index  $\alpha$  in (1.1.5) is 1. Let  $\gamma_{k,m,v}$  be the contraction number of the  $k$ th level V-cycle iteration with  $m$  pre-smoothing and  $m$  post-smoothing steps. According to Theorem 2.13, there is a constant  $C$ , independent of  $k$  and  $m$ , such that

$$m^{1/2}\gamma_{k,m,v} \leq C. \quad (5.5.1)$$

The numerical results in Table 5.1 are consistent with (5.5.1). In fact, they seem to indicate that  $C$  could be some number less than 10, and that (5.5.1) is valid for  $m \geq 50$ .

$m^{1/2}\gamma_{m,k,v}$	m=20	m=30	m=40	m=50	m=60	m=70	m=80
k=3	1.1347	1.0005	0.9075	0.8251	0.7454	0.6683	0.5948
k=4	1.5547	2.2969	2.1645	2.0767	2.0067	1.9455	1.8904
k=5	3.9972	3.5275	3.3043	3.1722	3.0803	3.0009	2.9491
k=6	5.3075	4.5923	4.2589	4.0665	3.9393	3.8459	3.7726
k=7	6.4352	5.4653	5.0207	4.7645	4.5984	4.4800	4.3898
k=8	7.3727	6.1637	5.6143	5.2982	5.0953	4.9528	4.8459

Table 5.1: Morley: V-cycle results on the unit square

**Remark 5.14** Note that the condition number of the operator  $A_k$  is of order  $h_k^{-4}$  (cf. (5.1.4) and (3.1.4)) while the condition number for second order problems is of order  $h_k^{-2}$ . Therefore the effect of  $m$  smoothing steps for fourth order problems is equivalent to the effect of  $\sqrt{m}$  smoothing steps for second order problems.

The key to the improvement of the performance of multigrid methods for fourth

order problems is in the design of new smoothing operators. Besides [6], [14] and [26], there are also some new composite relaxation schemes that may also apply (cf. [40]).

The design and analysis of more sophisticated smoothers would be one of the interesting research topics in the future. In [30], some smoothers that involve a multigrid solve for the Poisson problem have been obtained and turn out to be good ones.

Let  $\gamma_{k,m,f}$  be the contraction number of the  $k$ th level F-cycle iteration with  $m$  pre-smoothing and  $m$  post-smoothing steps. According to Theorem 2.14, there is a constant  $C$ , independent of  $k$  and  $m$ , such that

$$m^{1/2}\gamma_{k,m,f} \leq C. \tag{5.5.2}$$

The numerical results in Table 5.2 are consistent with (5.5.2) and seem to indicate that  $C = 2$  and (5.5.2) is valid as long as  $m \geq 15$ .

$m^{1/2}\gamma_{m,k,f}$	m=10	m=11	m=12	m=13	m=14	m=15	m=16
k=3	1.2132	1.1890	1.1706	1.1524	1.1364	1.1231	1.1071
k=4	1.4359	1.4037	1.3764	1.4097	1.3907	1.3838	1.4097
k=5	1.4310	1.3909	1.4163	1.4140	1.4030	1.4000	1.3933
k=6	1.4057	1.4041	1.3989	1.4017	1.3924	1.3908	1.3905
k=7	1.7918	1.3958	1.4035	1.3841	1.3756	1.3949	1.3759
k=8	5.4706	3.5578	2.4361	1.7541	1.3662	1.3775	1.3700

Table 5.2: Morley: F-cycle results on the unit square

In the case of the L-shaped domain (cf. Figure 5.10(b)), the index of elliptic regularity is  $\alpha_* = 0.5444837368$ . Numerical results for V-cycle and F-cycle algorithms are reported in Table 3 and Table 4, which are also consistent with (5.5.1) and (5.5.2).

**Remark 5.15** Even though the asymptotic convergence rate for both algorithms is  $O(m^{-\alpha/2})$ , the performance of the F-cycle algorithm is clearly superior, as demonstrated by the numerical results in Table 5.5 and Table 5.6. Similar results also hold for the L-shaped domain.

$m^{\alpha^*/2}\gamma_{m,k,v}$	m=30	m=40	m=50	m=60	m=70	m=80	m=90
k=3	0.2237	0.1404	0.0879	0.0546	0.0337	0.0208	0.0127
k=4	0.9099	0.7905	0.7137	0.6597	0.6173	0.6833	0.5561
k=5	1.5698	1.3924	1.2888	1.2124	1.1605	1.1192	1.0877
k=6	2.1111	1.8776	1.7316	1.6282	1.5592	1.5073	1.4635
k=7	2.5752	2.2753	2.0991	1.9814	1.8938	1.8276	1.7715
k=8	2.9648	2.6232	2.4243	2.2913	2.1894	2.1138	2.0517

Table 5.3: Morley: V-cycle results on an L-shaped domain

$m^{\alpha^*/2}\gamma_{m,k,f}$	m=11	m=12	m=13	m=14	m=15	m=16	m=17
k=3	0.5550	0.5266	0.5006	0.4762	0.4532	0.4313	0.4112
k=4	0.8529	0.8459	0.8126	0.8080	0.7965	0.7858	0.7743
k=5	0.8274	0.8092	0.7942	0.7701	0.7646	0.7303	0.7341
k=6	0.8134	0.7958	0.7830	0.7627	0.7515	0.7391	0.7192
k=7	0.8205	0.8038	0.7894	0.7759	0.7601	0.7381	0.7246
k=8	2.0406	1.4087	1.0198	0.7749	0.7449	0.7264	0.7140

Table 5.4: Morley: F-cycle results on an L-shaped domain

$\gamma_{m,k,v}$	m=34	m=35	m=36	m=37	m=38	m=39	m=40	m=41
k=3	0.1648	0.1609	0.1571	0.1535	0.1500	0.1467	0.1435	0.1404
k=4	0.3834	0.3756	0.3683	0.3613	0.3546	0.3483	0.3422	0.3365
k=5	0.5869	0.5746	0.5630	0.5521	0.5417	0.5318	0.5225	0.5135
k=6	0.7605	0.7438	0.7281	0.7133	0.6993	0.6860	0.6734	0.6614
k=7	0.9014	0.8807	0.8613	0.8430	0.8257	0.8093	0.7935	0.7791
k=8	1.0128	0.9887	0.9667	0.9448	0.9247	0.9057	0.8877	0.8707

Table 5.5: Morley: Contraction numbers for V-cycle algorithms on the unit square

Compared with the W-cycle algorithm, the contraction numbers for F-cycle are larger for small numbers of smoothing steps. In Table 5.7, the contraction numbers of W-cycle algorithm are given for  $3 \leq m \leq 8$ . In general F-cycle algorithms diverge for these  $m$ 's. However, for larger  $m$  (say, for  $m \geq 13$ ), the contraction numbers for both algorithms are almost the same, and sometimes the F-cycle algorithm is even better (cf. Table 5.6 and Table 5.8). Considering the fact that the cost for W-cycle is higher, we could say that the F-cycle algorithm is even more efficient than the W-

$\gamma_{m,k,f}$	m=11	m=12	m=13	m=14	m=15	m=16
k=3	0.3580	0.3379	0.3196	0.3037	0.2900	0.2768
k=4	0.4232	0.3973	0.3910	0.3717	0.3573	0.3524
k=5	0.4194	0.4089	0.3922	0.3750	0.3615	0.3483
k=6	0.4234	0.4038	0.3888	0.3721	0.3591	0.3476
k=7	0.4208	0.4051	0.3839	0.3677	0.3602	0.3440
k=8	1.0727	0.7032	0.4865	0.3651	0.3557	0.3425

Table 5.6: Morley: Contraction numbers for F-cycle algorithms on the unit square cycle algorithm for  $m$  between 11 and 16. It would be interesting to find a theoretical explanation for the superior performance of the F-cycle algorithm (see also [49]).

$\gamma_{m,k,w}$	m=3	m=4	m=5	m=6	k=7	m=8
k=3	0.7260	0.6620	0.5922	0.5313	0.4883	0.4245
k=4	0.7628	0.7005	0.6294	0.5787	0.5323	0.4898
k=5	0.8499	0.7477	0.6505	0.5743	0.5348	0.4988
k=6	0.8926	0.7673	0.6660	0.5850	0.5445	0.4990
k=7	0.9349	0.8051	0.6544	0.5874	0.5384	0.5003
k=8	0.9334	0.8214	0.6747	0.5856	0.5362	0.5009

Table 5.7: Morley: Contraction numbers for W-cycle algorithms on the unit square

$\gamma_{m,k,w}$	m=11	m=12	m=13	m=14	m=15	m=16
k=4	0.4250	0.4113	0.3943	0.3790	0.3660	0.3473
k=5	0.4288	0.4078	0.3958	0.3774	0.3670	0.3558
k=6	0.4296	0.4137	0.3957	0.3803	0.3642	0.3553
k=7	0.4285	0.4117	0.3954	0.3817	0.3667	0.3546

Table 5.8: Morley: Contraction numbers for W-cycle algorithms for large  $m$ 's.

# Chapter 6

## Multigrid Methods Based on the Incomplete Biquadratic Element

### 6.1 The Incomplete Biquadratic Element

The incomplete biquadratic element is defined on a rectangle. The set of shape functions is

$$\mathcal{P}_I = \langle 1, x_1, x_2, x_1x_2, x_1^2, x_2^2, x_1^2x_2, x_1x_2^2 \rangle.$$

The nodal variables include the evaluations of the functions at the vertices of the rectangles, and the normal derivatives at the midpoints of the edges (cf. Figure 6.1).

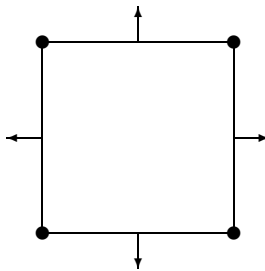


Figure 6.1: The incomplete biquadratic element

Let  $\mathcal{T}_1$  be a triangulation of the domain  $\Omega$  consisting of rectangles. For  $k \geq 1$ , we obtain  $\mathcal{T}_{k+1}$  by connecting the midpoints of the opposite edges of the rectangles in  $\mathcal{T}_k$ .

The incomplete biquadratic element space  $V_k$  consists of all functions  $v \in L_2(\Omega)$  satisfying the following conditions:

1.  $v_T \in \mathcal{P}_I$  for all  $T \in \mathcal{T}_k$ .
2.  $v$  is continuous at the vertices of  $\mathcal{T}_k$  and vanishes at the vertices along  $\partial\Omega$ .
3. The normal derivative  $\partial v/\partial n$  is continuous at the midpoints of interelement boundaries and vanishes at the midpoints along  $\partial\Omega$ .

The incomplete biquadratic finite element method for the biharmonic problem is as follows:

Find  $u_k \in V_k$  so that

$$a_k(u_k, v) = \int_{\Omega} f v \, dx \quad \forall v \in V_k.$$

A discrete inner product  $(\cdot, \cdot)_k$  on  $V_k$  is defined as follows:

$$(v_1, v_2)_k := h_k^2 \left[ \sum_{p \in \mathcal{V}_k} n(p) v_1(p) v_2(p) + \sum_{e \in \mathcal{E}_k} \left( \int_e \frac{\partial v_1}{\partial n} \, ds \right) \left( \int_e \frac{\partial v_2}{\partial n} \, ds \right) \right],$$

where  $\mathcal{V}_k$  is the set of internal vertices of  $\mathcal{T}_k$ ,  $\mathcal{E}_k$  is the set of internal edges of  $\mathcal{T}_k$  and  $n(p) = \frac{1}{4} \times$  (the number of quadrilaterals sharing the node  $p$  as a vertex).

The coarse-to-fine intergrid transfer operator  $I_{k-1}^k : V_{k-1} \rightarrow V_k$  is defined as follows:

Let  $v \in V_{k-1}$ . We define  $I_{k-1}^k v \in V_k$  by the following averaging technique :

1. If  $p$  is an internal vertex of  $\mathcal{T}_k$ , then

$$(I_{k-1}^k v)(p) = \frac{1}{|S_{p,k-1}|} \sum_{T \in S_{p,k-1}} v_T(p),$$

where  $S_{p,k-1} := \{T \in \mathcal{T}_{k-1} : p \in \partial T\}$ .

2. If  $e$  is an internal edge of  $\mathcal{T}_k$ , which means that  $e \subset \partial T_1 \cap \partial T_2$  for some  $T_1, T_2 \in \mathcal{T}_k$ , then

$$\int_e \frac{\partial (I_{k-1}^k v)}{\partial n} \, ds = \frac{1}{2} \left( \int_e \frac{\partial v_{T_1}}{\partial n} \, ds + \int_e \frac{\partial v_{T_2}}{\partial n} \, ds \right).$$

The fine-to-coarse operator  $I_k^{k-1} : V_k \longrightarrow V_{k-1}$  is the transpose of  $I_{k-1}^k$  with respect to the discrete inner products.

## 6.2 Convergence Analysis

A conforming relative of the incomplete biquadratic element (for rectangular mesh) is a generalized Bogner-Fox-Schmit  $\mathbb{Q}_4$  element (cf. [29]) described as follows.

The set of the shape functions for the element is

$$\mathbb{Q}_4 = \left\{ \sum_j p_j(x_1)q_j(x_2) : p_j \text{ and } q_j \text{ are polynomials of degree } \leq 4 \right\}.$$

The nodal variables include the evaluations of the shape functions at the vertices of the rectangle, at the center of the rectangle, and at the midpoints of the edges, the gradients at the vertices, mixed second order derivatives at the vertices, and the normal derivatives at the midpoints of the edges (cf. Figure 6.2).

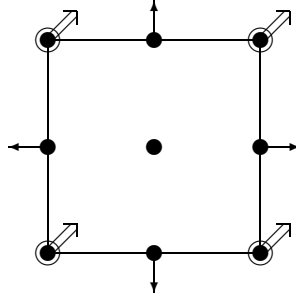


Figure 6.2: A generalized Bogner-Fox-Schmit  $\mathbb{Q}_4$  element

The finite element space  $\tilde{V}_k$  is defined as follows: A function  $v$  belongs to  $\tilde{V}_k$  if and only if the following conditions are satisfied:

1.  $v \in C^1(\Omega) \cap H_0^2(\Omega)$ ,
2.  $v|_T \in \mathbb{Q}_4$  for all  $T \in \mathcal{T}_k$ ,
3. the mixed second order derivative  $\frac{\partial^2 v}{\partial x_1 \partial x_2}$  is continuous at the vertices of  $\mathcal{T}_k$ .

The operator  $E_k : V_k \longrightarrow \tilde{V}_k$  is defined by the averaging technique we used for the Morley element and the HCT element. The definition of  $F_k : \tilde{V}_k \longrightarrow V_k$  is straightforward.

Each function  $v \in V_k$  has the following properties:

1.  $v|_e$  is a quadratic polynomial for all edge  $e \in \mathcal{T}_k$ ,
2.  $\nabla v$  is continuous at the midpoints of the edges of  $\mathcal{T}_k$ .

These properties also hold for the Morley functions, and we can see from Chapter 5 that they are the key ingredients for the verifications of Assumptions 1–11 in Chapter 4. Therefore we can verify these assumptions for the incomplete biquadratic element in the same way.

### 6.3 Numerical Results

Again, we test the algorithms on the unit square and an L-shaped domain. We first look at the results on the unit square domain.

The contraction numbers for V-cycle, F-cycle and W-cycle algorithms are given in Tables 6.1–6.3.

$\gamma_{m,k,v}$	m=2	m=3	m=4	m=5	m=6	m=7	m=8
k=3	0.78	0.70	0.65	0.59	0.55	0.51	0.46
k=4	1.18	0.72	0.66	0.61	0.57	0.53	0.49
k=5	1.43	0.73	0.68	0.62	0.57	0.52	0.49
k=6	1.52	0.74	0.68	0.63	0.58	0.53	0.51
k=7	1.52	0.74	0.68	0.63	0.58	0.53	0.50
k=8	1.52	0.74	0.68	0.66	0.58	0.53	0.50

Table 6.1: Incomplete Biquadratic: V-cycle contraction numbers on the unit square

We can see that V-cycle algorithm converges for  $m \geq 3$ , and F-cycle and W-cycle algorithms converge for  $m \geq 2$ , which is much better than the algorithms using

$\gamma_{m,k,f}$	m=1	m=2	m=3	m=4	m=5	m=6	m=7
k=3	0.87	0.78	0.71	0.65	0.59	0.55	0.51
k=4	2.56	0.81	0.73	0.67	0.62	0.58	0.53
k=5	7.84	0.80	0.74	0.68	0.62	0.58	0.53
k=6	46.6	0.80	0.74	0.68	0.63	0.58	0.53
k=7	361	0.80	0.74	0.68	0.63	0.58	0.53
k=8	3555	0.80	0.74	0.68	0.63	0.58	0.53

Table 6.2: Incomplete Biquadratic: F-cycle contraction numbers on the unit square

$\gamma_{m,k,w}$	m=1	m=2	m=3	m=4	m=5	m=6	m=7
k=3	0.86	0.77	0.71	0.65	0.60	0.55	0.51
k=4	0.95	0.79	0.72	0.66	0.61	0.57	0.53
k=5	1.05	0.79	0.73	0.68	0.62	0.58	0.53
k=6	1.08	0.80	0.73	0.67	0.62	0.57	0.53
k=7	1.07	0.80	0.73	0.68	0.62	0.58	0.53
k=8	1.08	0.80	0.73	0.68	0.62	0.57	0.53

Table 6.3: Incomplete Biquadratic: W-cycle contraction numbers on the unit square

the Morley method. When convergent, they are even more efficient than the HCT method.

$m^{1/2} * \gamma_{m,k,v}$	m=3	m=4	m=5	m=6	m=7	m=8
k=3	1.22	1.29	1.32	1.35	1.35	1.30
k=4	1.26	1.32	1.37	1.39	1.40	1.39
k=5	1.26	1.35	1.39	1.40	1.38	1.38
k=6	1.27	1.35	1.40	1.41	1.41	1.42
k=7	1.28	1.36	1.40	1.42	1.41	1.41
k=8	1.28	1.36	1.40	1.41	1.41	1.40

Table 6.4: Incomplete Biquadratic: V-cycle results on the unit square

From Tables 6.4 and 6.5 we can see that the numerical results are consistent with Theorems 2.13 and 2.14.

The results on the L-shaped domain are given in Tables 6.6–6.8.

$m^{1/2} * \gamma_{m,k,f}$	m=2	m=3	m=4	m=5	m=6	m=7
k=3	1.10	1.22	1.30	1.31	1.36	1.36
k=4	1.14	1.27	1.34	1.40	1.41	1.39
k=5	1.13	1.28	1.36	1.39	1.42	1.41
k=6	1.14	1.28	1.36	1.40	1.42	1.41
k=7	1.14	1.28	1.36	1.41	1.42	1.41
k=8	1.14	1.28	1.36	1.41	1.42	1.41

Table 6.5: Incomplete Biquadratic: F-cycle results on the unit square

$m^{\alpha*/2} * \gamma_{m,k,v}$	m=3	m=4	m=5	m=6	m=7	m=8
k=3	0.89	0.87	0.84	0.81	0.79	0.75
k=4	0.95	0.93	0.90	0.87	0.81	0.80
k=5	0.96	0.94	0.92	0.87	0.84	0.80
k=6	0.97	0.95	0.91	0.87	0.84	0.81
k=7	0.97	0.95	0.91	0.87	0.87	0.88
k=8	0.97	0.95	0.89	0.87	0.86	0.88

Table 6.6: Incomplete Biquadratic: V-cycle results on an L-shaped domain

$m^{\alpha*/2} * \gamma_{m,k,f}$	m=2	m=3	m=4	m=5	m=6	m=7
k=3	0.89	0.88	0.83	0.85	0.82	0.79
k=4	0.94	0.95	0.92	0.89	0.86	0.83
k=5	0.95	0.95	0.94	0.91	0.86	0.82
k=6	0.95	0.96	0.95	0.92	0.88	0.84
k=7	0.95	0.96	0.95	0.92	0.88	0.83
k=8	0.95	0.97	0.95	0.92	0.88	0.83

Table 6.7: Incomplete Biquadratic: F-cycle results on an L-shaped domain

$m^{\alpha*/2} * \gamma_{m,k,w}$	m=2	m=3	m=4	m=5	m=6	m=7
k=3	0.88	0.87	0.84	0.80	0.71	0.79
k=4	0.93	0.93	0.93	0.90	0.86	0.82
k=5	0.94	0.96	0.92	0.90	0.88	0.82
k=6	0.94	0.96	0.94	0.91	0.87	0.83
k=7	0.95	0.96	0.94	0.91	0.87	0.82
k=8	0.95	0.96	0.94	0.91	0.87	0.83

Table 6.8: Incomplete Biquadratic: W-cycle results on an L-shaped domain

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