Discontinuous Versions of Some Classical Inequalities

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Outline

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- Classical and Discontinuous Poincaré-Friedrichs Inequalities

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- Classical and Discontinuous Korn's Inequalities

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Poincaré-Friedrichs inequalities for piecewise H^1 functions
SINUM **41** (2003), 306–324

Korn's inequalities for piecewise H^1 vector fields
Math. Comp. **73** (2004), 1067–1087

Classical Poincaré-Friedrichs Inequalities

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Ω is a bounded polyhedral domain in $\mathbb{R}^d$ ($d = 2, 3$).

$$\|u\|_{L_2(\Omega)} \leq C\Big(|u|_{H^1(\Omega)} + \Big|\int_\Gamma u\,ds\Big|\Big)$$

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for all $u \in H^1(\Omega)$ $\quad$ ($\Gamma \subseteq \partial\Omega$, $|\Gamma| > 0$)

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for all $u \in H^1(\Omega)$ $\quad$ ($\Gamma \subseteq \partial\Omega$, $|\Gamma| > 0$)

$H^1(\Omega) \subset\subset L_2(\Omega)$ (Rellich-Kondrachov)

Proof of a Poincaré-Friedrichs Inequality

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$$\|u\|_{L_2(\Omega)} \leq C\Big(|u|_{H^1(\Omega)} + \Big|\int_\Omega u\,dx\Big|\Big) \quad \forall\, u \in H^1(\Omega)$$

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Suppose this is not true.

Proof of a Poincaré-Friedrichs Inequality

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$\exists\, u_n \in H^1(\Omega)$ such that

$$\|u_n\|_{L_2(\Omega)} > n\Big(|u_n|_{H^1(\Omega)} + \Big|\int_\Omega u_n\,dx\Big|\Big)$$

Proof of a Poincaré-Friedrichs Inequality

$$\|u\|_{L_2(\Omega)} \leq C\Big(|u|_{H^1(\Omega)} + \Big|\int_\Omega u\,dx\Big|\Big) \quad \forall\, u \in H^1(\Omega)$$

$\exists\, u_n \in H^1(\Omega)$ such that $\|u_n\|_{L_2(\Omega)} = 1$ and

$$1 > n\Big(|u_n|_{H^1(\Omega)} + \Big|\int_\Omega u_n\,dx\Big|\Big)$$

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$\{u_n\}_{n=1}^{\infty}$ is a bounded sequence in $H^1(\Omega)$

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it follows from $H^1(\Omega) \subset\subset L_2(\Omega)$ that

$$\lim_{n\to\infty} u_n = u \quad \text{in } L_2(\Omega)$$

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in particular,

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therefore,

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in particular,

$$\lim_{n\to\infty} |u_n|_{H^1(\Omega)} = 0 \quad \text{and} \quad \lim_{n\to\infty} \int_\Omega u_n\,dx = 0$$

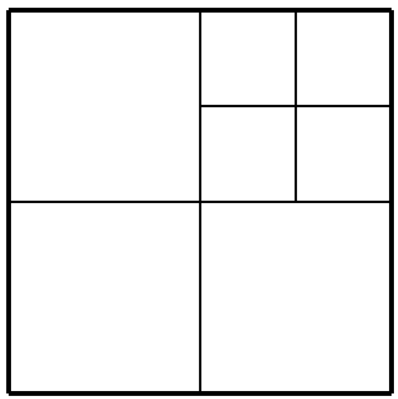
a contradiction

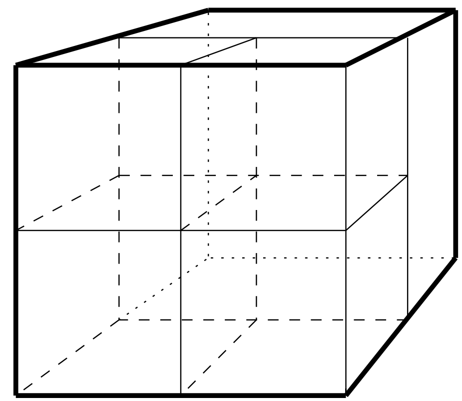
$$\|u\|_{L_2(\Omega)} = 1, \quad |u|_{H^1(\Omega)} = 0, \quad \int_\Omega u\,dx = 0$$

Discontinuous Poincaré-Friedrichs Inequalities

Discontinuous Poincaré-Friedrichs Inequalities

$\mathcal{P}$ is a partition of Ω.





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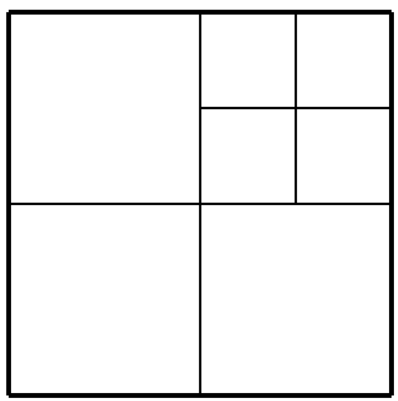

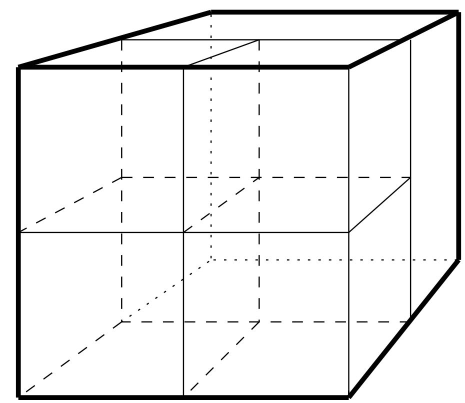

$$H^1(\Omega,\mathcal{P}) = \{u \in L_2(\Omega) : u_D = u\big|_D \in H^1(D) \quad \forall\, D \in \mathcal{P}\}$$

$S(\mathcal{P},\Omega) =$ the set of interior sides of $\mathcal{P}$

Discontinuous Poincaré-Friedrichs Inequalities

$$\|u\|^2_{L_2(\Omega)} \leq C\Big(|u|^2_{H^1(\Omega,\mathcal{P})} + \Big(\int_\Gamma u\,ds\Big)^2 + \mathfrak{J}\Big)$$

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$$\mathfrak{J} = \sum_{\sigma\in S(\mathcal{P},\Omega)} (\,\text{diam}\,\sigma\,)^{-1}\|\pi_{\sigma,0}[u]_\sigma\|^2_{L_2(\sigma)}$$

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$$|u|^2_{H^1(\Omega,\mathcal{P})} = \sum_{D\in\mathcal{P}} |u|^2_{H^1(D)}$$

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$[u]_\sigma =$ the jump of u across σ

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$\pi_{\sigma,0} =$ the orthogonal projection operator from $L_2(\sigma)$ onto $P_0(\sigma)$

Discontinuous Poincaré-Friedrichs Inequalities

$$\|u\|_{L_2(\Omega)}^2 \leq C\Big(|u|_{H^1(\Omega,\mathcal{P})}^2 + \Big(\int_\Gamma u\,ds\Big)^2 + \mathfrak{J}\Big)$$

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C depends only on the shape regularity of the partition.

Shape Regularity of a Partition

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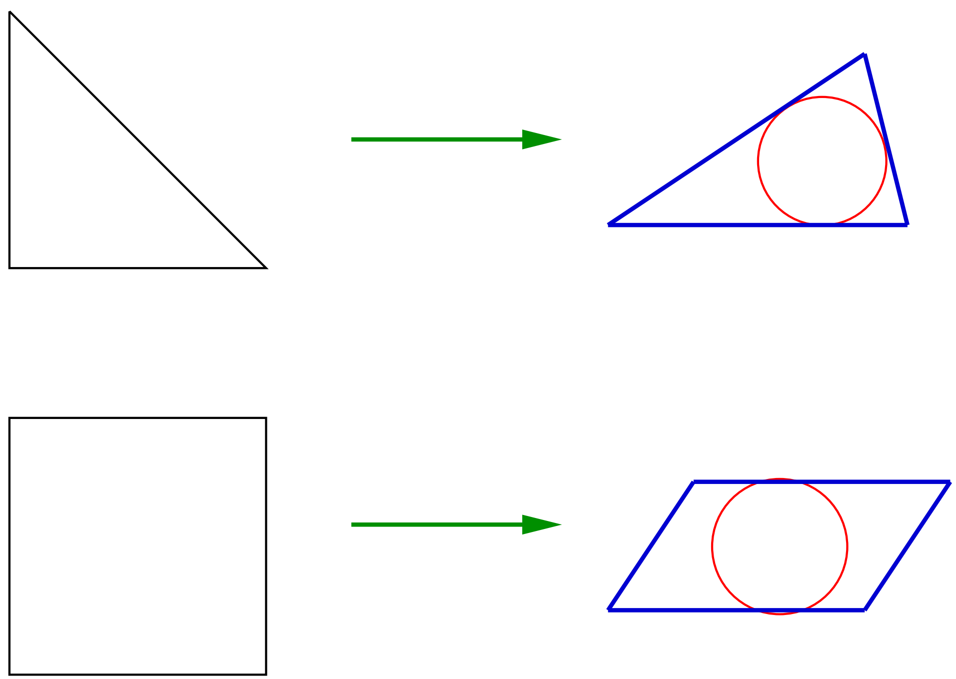
- shape regularity of each subdomain

Shape Regularity of a Partition

- shape regularity of each subdomain
  - reference domain and aspect ratio

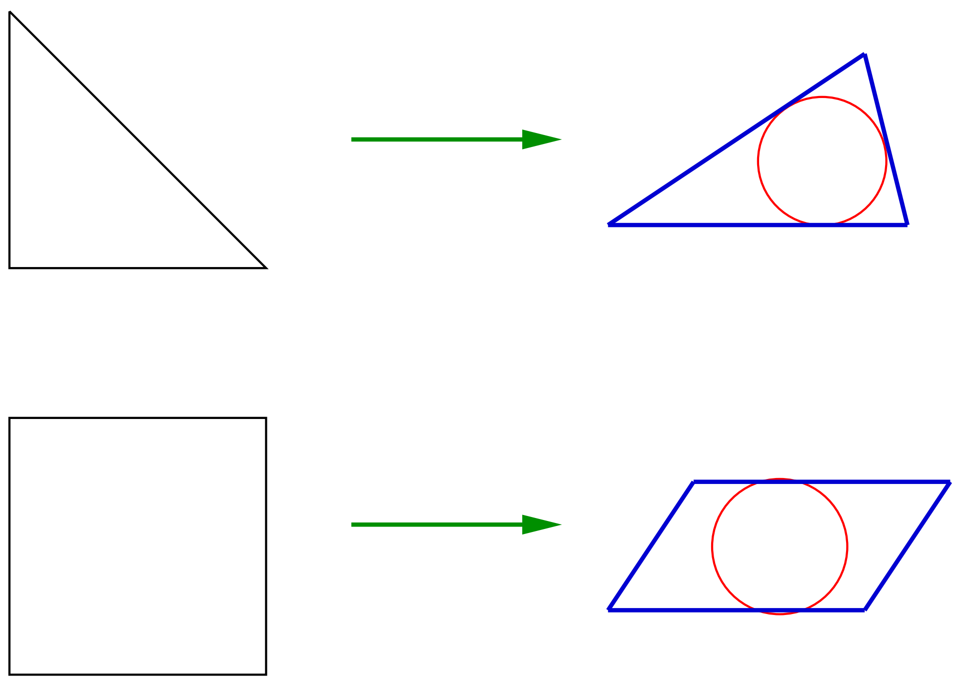
Shape Regularity of an Individual Subdomain

reference domain



Shape Regularity of an Individual Subdomain

reference domain

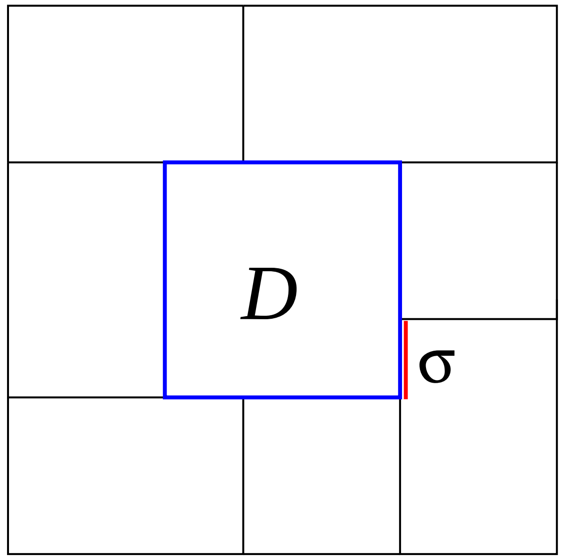


aspect ratio $=$ diam D/diam B

Shape Regularity of a Partition

- shape regularity of each subdomain
  - reference domain and aspect ratio
- relative positions of subdomains sharing a common side

Relative Positions of Neighboring Subdomains



This can be quantified by the ratios of the length of the boundary of a subdomain to the lengths of the edges on the boundary:

$$|\partial D|/|\sigma|$$

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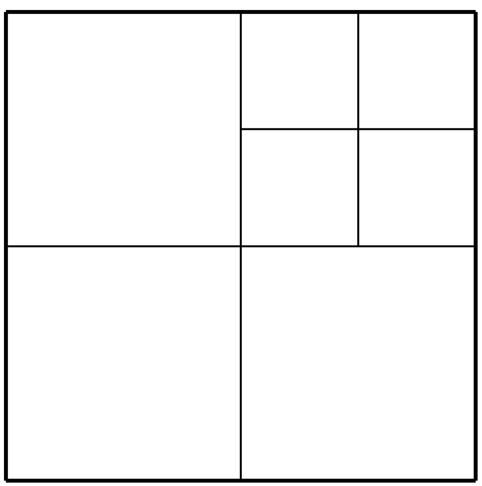

reference domain $=$ square

aspect ratio $= \sqrt{2}$

$$\frac{|\partial D|}{|\sigma|} \leq 8$$

Shape Regularity of a Partition

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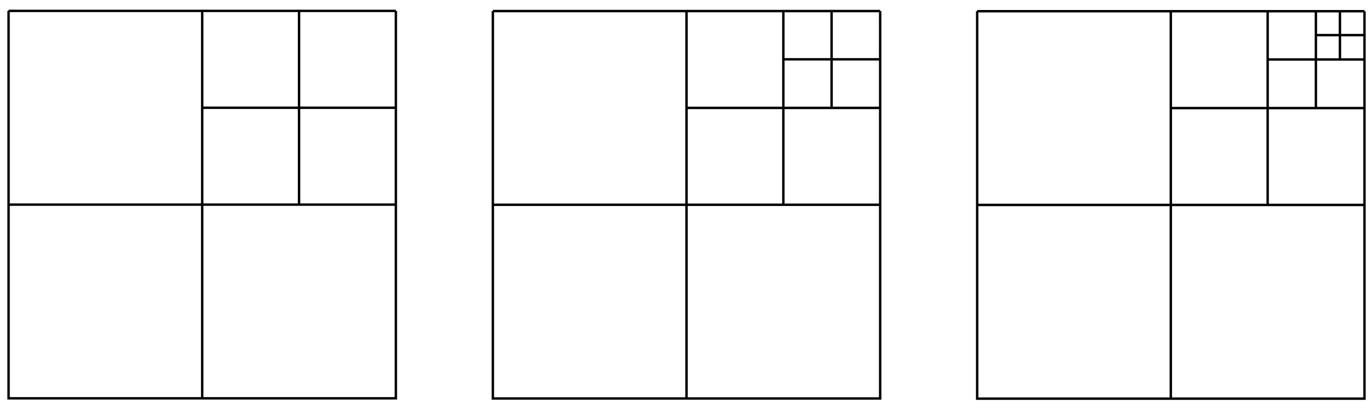

a uniform constant C for such partitions

Examples of Partitions

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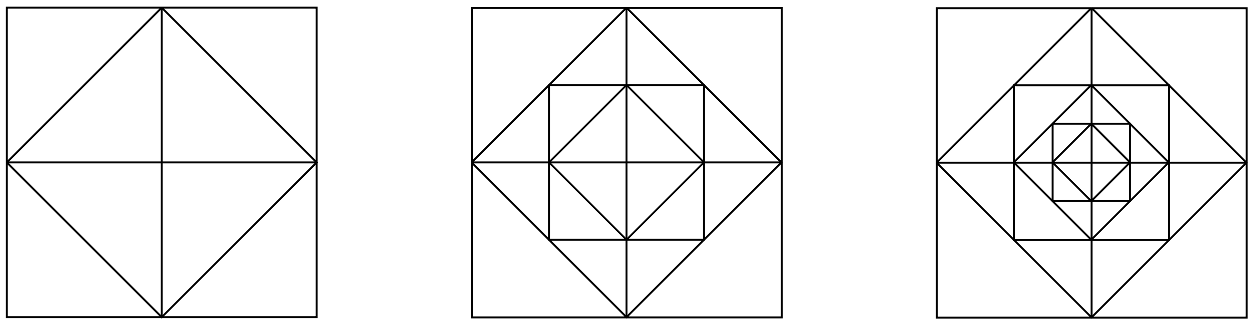
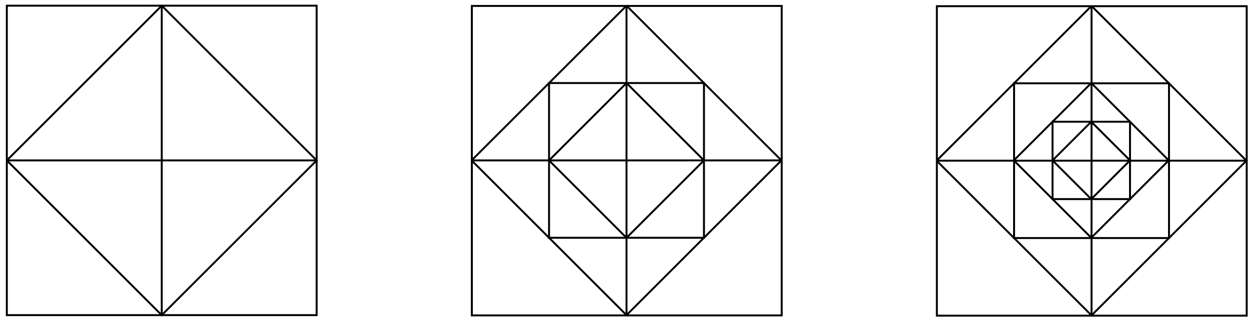
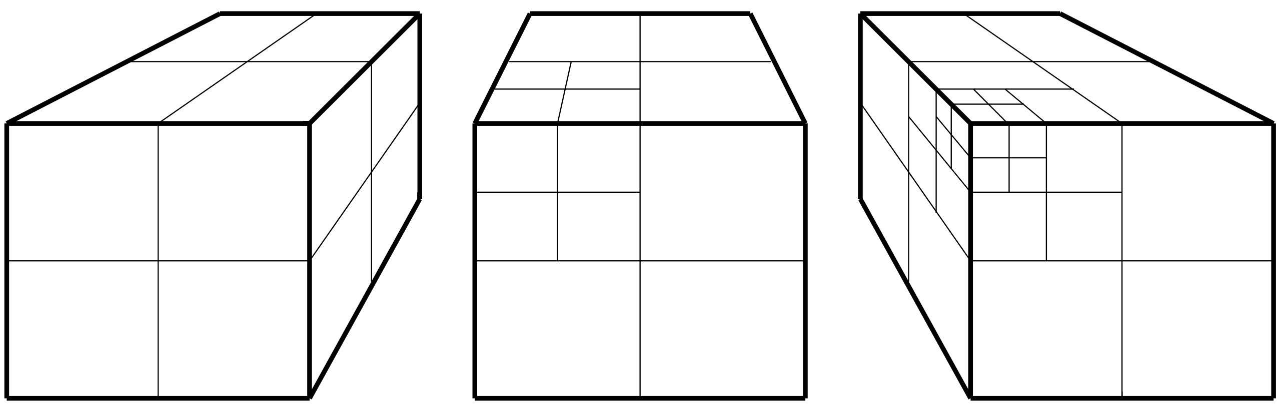


Examples of Partitions





Application to Nonconforming Finite Elements

$$\|u\|^2_{L_2(\Omega)} \leq C\Big(|u|^2_{H^1(\Omega,\mathcal{P})} + \Big(\int_\Gamma u\,ds\Big)^2 + \mathfrak{J}\Big)$$

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$$\mathfrak{J} = \sum_{\sigma\in S(\mathcal{P},\Omega)} (\,\text{diam}\,\sigma)^{-1} \|\pi_{\sigma,0}[u]_\sigma\|^2_{L_2(\sigma)}$$

$\pi_{\sigma,0}$ = the orthogonal projection operator from $L_2(\sigma)$ onto $P_0(\sigma)$

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$$\pi_{\sigma,0}[u]_\sigma = \frac{1}{|\sigma|}\int_\sigma [u]_\sigma\,ds$$

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$$\pi_{\sigma,0}[u]_\sigma = \frac{1}{|\sigma|}\int_\sigma [u]_\sigma\,ds$$

These inequalities can be simplified if $\int_\sigma [u]_\sigma\,ds = 0$.

Application to Nonconforming Finite Elements

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provided

$$\pi_{\sigma,0}[u]_\sigma = 0 \iff \int_\sigma [u]\,ds = 0 \qquad \forall\,\sigma \in S(\mathcal{P},\Omega)$$

This weak continuity condition is satisfied by many classical nonconforming finite element functions when $\mathcal{P}$ is a triangulation.

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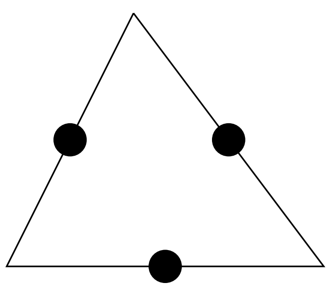
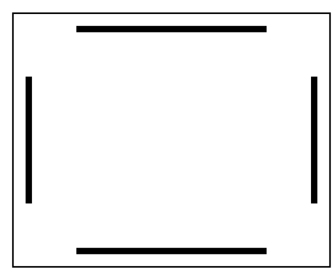
classical nonconforming finite element functions



$1, x, y$



$1, x, y, x^2 - y^2$

Temam 1977, Thomas 1977,...

Application to Mortar Methods

$$\|u\|_{L_2(\Omega)} \le C\Big(|u|_{H^1(\Omega,\mathcal{P})} + \Big|\int_\Gamma u\,ds\Big|\Big)$$

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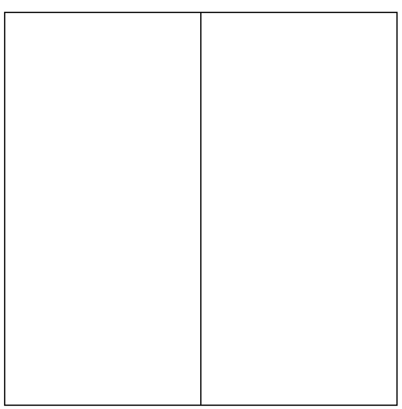

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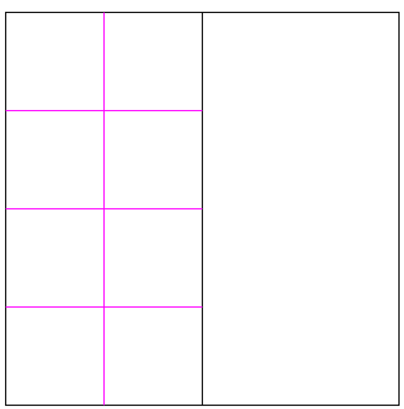

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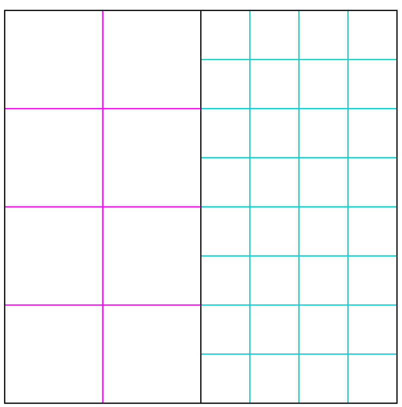

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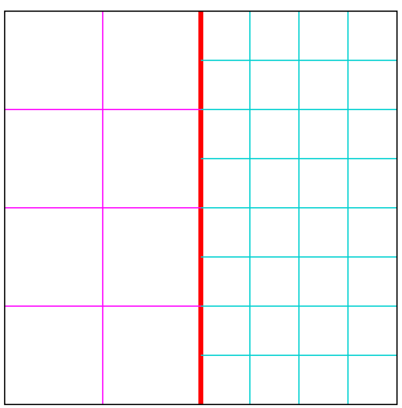

non-matching grids

discontinuity of mortar element

functions at the interface

Application to Mortar Methods

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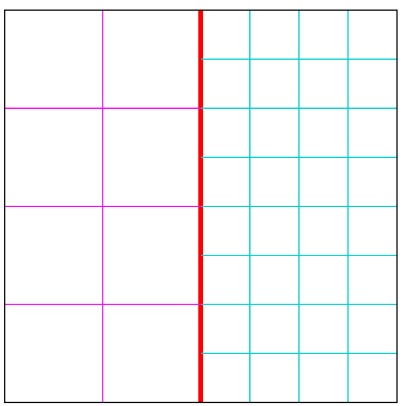

mortar condition on $[u]_\sigma$

implies

$$\int_\sigma [u]_\sigma \, ds = 0$$

2D - Stefanica 1998, Gopalakrishnan 1999, 2000

Application to Discontinuous Galerkin Methods

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2D - Arnold 1982

Arnold's Proof

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Let Ω be a convex polygon, $u \in H^1(\Omega, \mathcal{T})$ and

$$-\Delta\phi = u \quad \text{on } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega.$$

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$\phi \in H^2(\Omega) \cap H^1_0(\Omega)$ and $\|\phi\|_{H^2(\Omega)} \le C_\Omega \|u\|_{L_2(\Omega)}$

Arnold's Proof

Let Ω be a convex polygon, $u \in H^1(\Omega, \mathcal{T})$ and

$$-\Delta\phi = u \quad \text{on } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega.$$

$$\phi \in H^2(\Omega) \cap H^1_0(\Omega) \quad \text{and} \quad \|\phi\|_{H^2(\Omega)} \leq C_\Omega \|u\|_{L_2(\Omega)}$$

$$\|u\|^2_{L_2(\Omega)} = \sum_{T \in \mathcal{T}} \int_T u(-\Delta\phi)\, dx$$

Arnold's Proof

Let Ω be a convex polygon, $u \in H^1(\Omega, \mathcal{T})$ and

$$-\Delta\phi = u \quad \text{on } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega.$$

$\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\|\phi\|_{H^2(\Omega)} \leq C_\Omega \|u\|_{L_2(\Omega)}$

$$\begin{aligned}
\|u\|_{L_2(\Omega)}^2 &= \sum_{T \in \mathcal{T}} \int_T u(-\Delta\phi)\,dx \\
&= \sum_{T \in \mathcal{T}} \Big[-\int_{\partial T} u \frac{\partial\phi}{\partial n}\,ds + \int_T \nabla u \cdot \nabla\phi\,dx \Big]
\end{aligned}$$

Arnold's Proof

Let Ω be a convex polygon, $u \in H^1(\Omega, \mathcal{T})$ and

$$-\Delta\phi = u \quad \text{on } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega.$$

$\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\|\phi\|_{H^2(\Omega)} \leq C_\Omega \|u\|_{L_2(\Omega)}$

$$\begin{aligned}
\|u\|_{L_2(\Omega)}^2 &= \sum_{T\in\mathcal{T}} \int_T u(-\Delta\phi)\,dx \\
&= \sum_{e\in\mathcal{E}^i} \int_e [u]_e \frac{\partial\phi}{\partial n}\,ds - \sum_{e\in\mathcal{E}^b} \int_e u \frac{\partial\phi}{\partial n}\,ds \\
&\qquad + \sum_{T\in\mathcal{T}} \int_T \nabla u \cdot \nabla\phi\,dx
\end{aligned}$$

Arnold's Proof

Let Ω be a convex polygon, $u \in H^1(\Omega, \mathcal{T})$ and

$$-\Delta\phi = u \quad \text{on } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega.$$

$$\phi \in H^2(\Omega) \cap H^1_0(\Omega) \quad \text{and} \quad \|\phi\|_{H^2(\Omega)} \leq C_\Omega \|u\|_{L_2(\Omega)}$$

$$\begin{aligned} \|u\|^2_{L_2(\Omega)} \leq \sum_{e \in \mathcal{E}^i} &\|[u]_e\|_{L_2(e)} \|\partial\phi/\partial n\|_{L_2(e)} \\ &+ \sum_{e \in \mathcal{E}^b} \|u\|_{L_2(e)} \|\partial\phi/\partial n\|_{L_2(e)} \\ &+ \sum_{T \in \mathcal{T}} |u|_{H^1(T)} |\phi|_{H^1(T)} \end{aligned}$$

Arnold's Proof

Let Ω be a convex polygon, $u \in H^1(\Omega, \mathcal{T})$ and

$$-\Delta\phi = u \quad \text{on } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega.$$

$\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\|\phi\|_{H^2(\Omega)} \le C_\Omega \|u\|_{L_2(\Omega)}$

$$\|u\|_{L_2(\Omega)}^2 \le \frac{1}{2\epsilon} \sum_{e \in \mathcal{E}^i} |e|^{-1} \|[u]_e\|_{L_2(e)}^2 + \frac{\epsilon}{2} \sum_{e \in \mathcal{E}^i} |e| \, \|\partial\phi/\partial n\|_{L_2(e)}^2$$
$$+ \sum_{e \in \mathcal{E}^b} \|u\|_{L_2(e)} \|\partial\phi/\partial n\|_{L_2(e)}$$
$$+ \sum_{T \in \mathcal{T}} |u|_{H^1(T)} |\phi|_{H^1(T)}$$

Arnold's Proof

Let Ω be a convex polygon, $u \in H^1(\Omega, \mathcal{T})$ and

$$-\Delta\phi = u \quad \text{on } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega.$$

$\phi \in H^2(\Omega) \cap H^1_0(\Omega)$ and $\|\phi\|_{H^2(\Omega)} \leq C_\Omega \|u\|_{L_2(\Omega)}$

$$\begin{aligned}
\|u\|^2_{L_2(\Omega)} \leq{}& \frac{1}{2\epsilon} \sum_{e \in \mathcal{E}^i} |e|^{-1} \|[u]_e\|^2_{L_2(e)} + \frac{\epsilon}{2} \sum_{e \in \mathcal{E}^i} |e| \, \|\partial\phi/\partial n\|^2_{L_2(e)} \\
&+ \frac{1}{2\epsilon} \sum_{e \in \mathcal{E}^b} |e|^{-1} \|u\|^2_{L_2(e)} + \frac{\epsilon}{2} \sum_{e \in \mathcal{E}^b} |e| \, \|\partial\phi/\partial n\|^2_{L_2(e)} \\
&+ \sum_{T \in \mathcal{T}} |u|_{H^1(T)} |\phi|_{H^1(T)}
\end{aligned}$$

Arnold's Proof

Let Ω be a convex polygon, $u \in H^1(\Omega, \mathcal{T})$ and

$$-\Delta\phi = u \quad \text{on } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega.$$

$\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\|\phi\|_{H^2(\Omega)} \leq C_\Omega \|u\|_{L_2(\Omega)}$

$$\begin{aligned}
\|u\|^2_{L_2(\Omega)} \leq \frac{1}{2\epsilon} \sum_{e \in \mathcal{E}^i} |e|^{-1} \|[u]_e\|^2_{L_2(e)} + \frac{\epsilon}{2} \sum_{e \in \mathcal{E}^i} |e| \, \|\partial\phi/\partial n\|^2_{L_2(e)} \\
+ \frac{1}{2\epsilon} \sum_{e \in \mathcal{E}^b} |e|^{-1} \|u\|^2_{L_2(e)} + \frac{\epsilon}{2} \sum_{e \in \mathcal{E}^b} |e| \, \|\partial\phi/\partial n\|^2_{L_2(e)} \\
+ \frac{1}{2\epsilon} |u|^2_{H^1(\Omega, \mathcal{T})} + \frac{\epsilon}{2} |\phi|^2_{H^1(\Omega)}
\end{aligned}$$

Arnold's Proof

Let Ω be a convex polygon, $u \in H^1(\Omega, \mathcal{T})$ and

$$-\Delta\phi = u \quad \text{on } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega.$$

$\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\|\phi\|_{H^2(\Omega)} \le C_\Omega \|u\|_{L_2(\Omega)}$

$$\begin{aligned}\|u\|_{L_2(\Omega)}^2 \le & \frac{1}{2\epsilon} \sum_{e \in \mathcal{E}^i} |e|^{-1} \|[u]_e\|_{L_2(e)}^2 + C\epsilon \|\phi\|_{H^2(\Omega)}^2 \\ & + \frac{1}{2\epsilon} \sum_{e \in \mathcal{E}^b} |e|^{-1} \|u\|_{L_2(e)}^2 + C\epsilon \|\phi\|_{H^2(\Omega)}^2 \\ & + \frac{1}{2\epsilon} |u|_{H^1(\Omega,\mathcal{T})}^2 + \frac{\epsilon}{2} |\phi|_{H^1(\Omega)}^2\end{aligned}$$

Arnold's Proof

Let Ω be a convex polygon, $u \in H^1(\Omega, \mathcal{T})$ and

$$-\Delta\phi = u \quad \text{on } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega.$$

$\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\|\phi\|_{H^2(\Omega)} \leq C_\Omega \|u\|_{L_2(\Omega)}$

$$\begin{aligned}
\|u\|_{L_2(\Omega)}^2 \leq \frac{1}{2\epsilon} \sum_{e \in \mathcal{E}^i} |e|^{-1} \|[u]_e\|_{L_2(e)}^2 + C\epsilon \|u\|_{L_2(\Omega)}^2 \\
+ \frac{1}{2\epsilon} \sum_{e \in \mathcal{E}^b} |e|^{-1} \|u\|_{L_2(e)}^2 + C\epsilon \|u\|_{L_2(\Omega)}^2 \\
+ \frac{1}{2\epsilon} |u|_{H^1(\Omega,\mathcal{T})}^2 + C\epsilon \|u\|_{L_2(\Omega)}^2
\end{aligned}$$

Arnold's Proof

Let Ω be a convex polygon, $u \in H^1(\Omega, \mathcal{T})$ and

$$-\Delta\phi = u \quad \text{on } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega.$$

$\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\|\phi\|_{H^2(\Omega)} \leq C_\Omega \|u\|_{L_2(\Omega)}$

$$\|u\|_{L_2(\Omega)}^2 \leq C\Big[\sum_{e\in\mathcal{E}^i} |e|^{-1} \|[u]_e\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}^b} |e|^{-1} \|u\|_{L_2(e)}^2 + |u|_{H^1(\Omega,\mathcal{T})}^2\Big]$$

Arnold's Proof

Let Ω be a convex polygon, $u \in H^1(\Omega, \mathcal{T})$ and

$$-\Delta\phi = u \quad \text{on } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega.$$

$\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\|\phi\|_{H^2(\Omega)} \le C_\Omega \|u\|_{L_2(\Omega)}$

$$\|u\|^2_{L_2(\Omega)} \le C\Big[\sum_{e\in\mathcal{E}^i} |e|^{-1}\|[u]_e\|^2_{L_2(e)} + \sum_{e\in\mathcal{E}^b} |e|^{-1}\|u\|^2_{L_2(e)} + |u|^2_{H^1(\Omega,\mathcal{T})}\Big]$$

- The integrals on the whole $\partial\Omega$ are penalized.

Arnold's Proof

Let Ω be a convex polygon, $u \in H^1(\Omega, \mathcal{T})$ and

$$-\Delta\phi = u \quad \text{on } \Omega, \qquad \phi = 0 \quad \text{on } \partial\Omega.$$

$\phi \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\|\phi\|_{H^2(\Omega)} \leq C_\Omega \|u\|_{L_2(\Omega)}$

$$\|u\|_{L_2(\Omega)}^2 \leq C\Big[\sum_{e\in\mathcal{E}^i} |e|^{-1}\|[u]_e\|_{L_2(e)}^2 + \sum_{e\in\mathcal{E}^b} |e|^{-1}\|u\|_{L_2(e)}^2 + |u|_{H^1(\Omega,\mathcal{T})}^2\Big]$$

- The integrals on the whole $\partial\Omega$ are penalized.
- The proof involves elliptic regularity theory.

Classical Korn's Inequalities

Classical Korn's Inequalities

Korn's inequality

$$|\boldsymbol{u}|_{H^1(\Omega)} \leq C_\Omega \Big(\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L_2(\Omega)} + \|\boldsymbol{u}\|_{L_2(\Omega)} \Big)$$

for all $\boldsymbol{u} \in [H^1(\Omega)]^d$

Classical Korn's Inequalities

Korn's inequality

$$|\boldsymbol{u}|_{H^1(\Omega)} \leq C_\Omega\Big(\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L_2(\Omega)} + \|\boldsymbol{u}\|_{L_2(\Omega)}\Big)$$

for all $\boldsymbol{u} \in [H^1(\Omega)]^d$

$$\epsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big) \quad \text{for} \quad 1 \leq i, j \leq d$$

Classical Korn's Inequalities

Korn's inequality

$$|\boldsymbol{u}|_{H^1(\Omega)} \leq C_\Omega \Big(\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L_2(\Omega)} + \|\boldsymbol{u}\|_{L_2(\Omega)} \Big)$$

for all $\boldsymbol{u} \in [H^1(\Omega)]^d$

Lions' Lemma

$\phi \in H^{-1}(\Omega)$ and $\nabla\phi \in [H^{-1}(\Omega)]^d$ implies $\phi \in L_2(\Omega)$

Classical Korn's Inequalities

Korn's inequality

$$|\boldsymbol{u}|_{H^1(\Omega)} \leq C_\Omega \Big(\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L_2(\Omega)} + \|\boldsymbol{u}\|_{L_2(\Omega)} \Big)$$

for all $\boldsymbol{u} \in [H^1(\Omega)]^d$

Lions' Lemma

$\phi \in H^{-1}(\Omega)$ and $\nabla\phi \in [H^{-1}(\Omega)]^d$ implies $\phi \in L_2(\Omega)$

$$\|\phi\|_{L_2(\Omega)} \leq C_\Omega \Big(\|\phi\|_{H^{-1}(\Omega)} + \|\nabla\phi\|_{H^{-1}(\Omega)} \Big)$$

Classical Korn's Inequalities

Korn's inequality

$$|\boldsymbol{u}|_{H^1(\Omega)} \le C_\Omega \Big(\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L_2(\Omega)} + \|\boldsymbol{u}\|_{L_2(\Omega)} \Big)$$

Korn's first inequality

$$|\boldsymbol{u}|_{H^1(\Omega)} \le C_{\Omega,\Gamma} \Big(\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L_2(\Omega)} + \|\pi_{\Gamma,1}\boldsymbol{u}\|_{L_2(\Gamma)} \Big)$$

Classical Korn's Inequalities

Korn's inequality

$$|\boldsymbol{u}|_{H^1(\Omega)} \leq C_\Omega\Big(\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L_2(\Omega)} + \|\boldsymbol{u}\|_{L_2(\Omega)}\Big)$$

Korn's first inequality

$$|\boldsymbol{u}|_{H^1(\Omega)} \leq C_{\Omega,\Gamma}\Big(\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L_2(\Omega)} + \|\pi_{\Gamma,1}\boldsymbol{u}\|_{L_2(\Gamma)}\Big)$$

$$\Gamma \subseteq \partial\Omega, \quad |\Gamma| > 0$$

$\pi_{\Gamma,1}$ = the orthogonal projection operator from

$[L_2(\Gamma)]^d$ onto $[P_1(\Gamma)]^d$

Classical Korn's Inequalities

Korn's second inequality

$$|\boldsymbol{u}|_{H^1(\Omega)} \leq C_\Omega \Big(\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L_2(\Omega)} + \Big| \int_\Omega \nabla \times \boldsymbol{u}\, dx \Big| \Big)$$

Classical Korn's Inequalities

Korn's second inequality

$$|\boldsymbol{u}|_{H^1(\Omega)} \le C_\Omega \Big(\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L_2(\Omega)} + \Big| \int_\Omega \nabla \times \boldsymbol{u}\, dx \Big| \Big)$$

$d = 3$ (curl of $\boldsymbol{u}$)

$$\nabla \times \boldsymbol{u} = \Big(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}, \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}, \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \Big)^t$$

Classical Korn's Inequalities

Korn's second inequality

$$|\boldsymbol{u}|_{H^1(\Omega)} \leq C_\Omega \Big(\|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L_2(\Omega)} + \Big| \int_\Omega \nabla \times \boldsymbol{u}\, dx \Big| \Big)$$

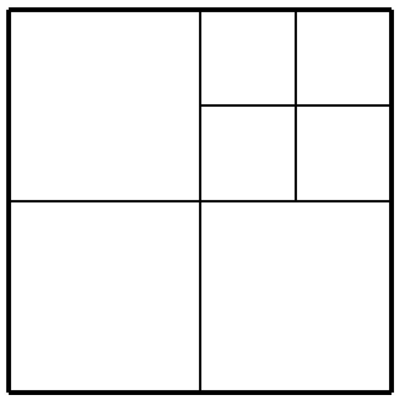
$d = 2$ (rot of $\boldsymbol{u}$)

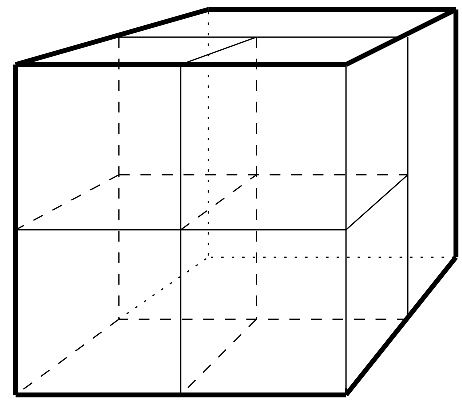
$$\nabla \times \boldsymbol{u} = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

Discontinuous Korn's Inequalities

Discontinuous Korn's Inequalities

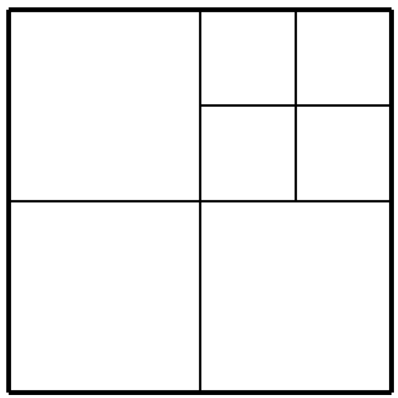
$\mathcal{P}$ is a partition of Ω.

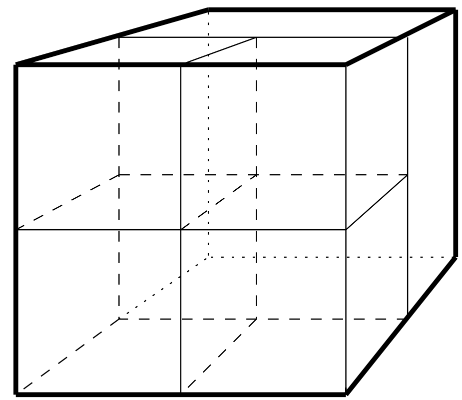




Discontinuous Korn's Inequalities

$\mathcal{P}$ is a partition of Ω.





$$[H^1(\Omega,\mathcal{P})]^d = \{\boldsymbol{u} \in [L_2(\Omega)]^d : u\big|_D \in [H^1(D)]^d \quad \forall\, D \in \mathcal{P}\}$$

$S(\mathcal{P},\Omega) =$ the set of interior sides of $\mathcal{P}$

Discontinuous Korn's Inequalities

Korn's inequality

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \|\boldsymbol{u}\|^2_{L_2(\Omega)} + \mathfrak{I} \Big)$$

$$\mathfrak{I} = \sum_{\sigma \in S(\mathcal{P},\Omega)} (\,\text{diam}\,\sigma)^{-1} \big\|\pi_{\sigma,1}[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)}$$

Discontinuous Korn's Inequalities

Korn's inequality

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \|\boldsymbol{u}\|^2_{L_2(\Omega)} + \mathfrak{J}\Big)$$

$$\mathfrak{J} = \sum_{\sigma\in S(\mathcal{P},\Omega)} (\,\text{diam}\,\sigma\,)^{-1} \big\|\pi_{\sigma,1}[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)}$$

$$\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{v})\big|_D = \boldsymbol{\epsilon}(\boldsymbol{v}\big|_D) \qquad \forall\, D \in \mathcal{P}$$

Discontinuous Korn's Inequalities

Korn's inequality

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \leq C\left(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \|\boldsymbol{u}\|^2_{L_2(\Omega)} + \mathfrak{I}\right)$$

$$\mathfrak{I} = \sum_{\sigma\in S(\mathcal{P},\Omega)} (\,\text{diam}\,\sigma)^{-1}\left\|\pi_{\sigma,1}[\boldsymbol{u}]_\sigma\right\|^2_{L_2(\sigma)}$$

$\pi_{\sigma,1}$ = the orthogonal projection operator from

$[L_2(\sigma)]^d$ onto $[P_1(\sigma)]^d$.

Discontinuous Korn's Inequalities

Korn's inequality

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \|\boldsymbol{u}\|^2_{L_2(\Omega)} + \mathfrak{J}\Big)$$

$$\mathfrak{J} = \sum_{\sigma \in S(\mathcal{P},\Omega)} (\,\text{diam}\,\sigma)^{-1} \big\|\pi_{\sigma,1}[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)}$$

C depends only on the shape regularity of the partition.

Discontinuous Korn's Inequalities

Korn's inequality

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \|\boldsymbol{u}\|^2_{L_2(\Omega)} + \mathfrak{J}\Big)$$

$$\mathfrak{J} = \sum_{\sigma\in S(\mathcal{P},\Omega)} (\,\text{diam}\,\sigma)^{-1} \Big\|\pi_{\sigma,1}[\boldsymbol{u}]_\sigma\Big\|^2_{L_2(\sigma)}$$

Korn's first inequality

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \|\pi_{\Gamma,1}\boldsymbol{u}\|^2_{L_2(\Gamma)} + \mathfrak{J}\Big)$$

Korn's second inequality

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \Big|\sum_{D\in\mathcal{P}} \int_D \nabla\times\boldsymbol{u}\,dx\Big|^2 + \mathfrak{J}\Big)$$

Discontinuous Korn's Inequalities

Korn's inequality

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \|\boldsymbol{u}\|^2_{L_2(\Omega)} + \mathfrak{J}\Big)$$

$$\mathfrak{J} = \sum_{\sigma\in S(\mathcal{P},\Omega)} \big(\,\text{diam}\,\sigma\big)^{-1}\Big\|\pi_{\sigma,1}[\boldsymbol{u}]_\sigma\Big\|^2_{L_2(\sigma)}$$

These Korn's inequalities can be simplified if

$$\pi_{\sigma,1}[\boldsymbol{u}]_\sigma = \mathbf{0}$$

Application to Nonconforming Finite Elements

Korn's inequality

$$|\boldsymbol{u}|_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|_{L_2(\Omega)} + \|\boldsymbol{u}\|_{L_2(\Omega)} \Big)$$

Korn's first inequality

$$|\boldsymbol{u}|_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|_{L_2(\Omega)} + \|\pi_{\Gamma,1}\boldsymbol{u}\|_{L_2(\Gamma)} \Big)$$

Korn's second inequality

$$|\boldsymbol{u}|_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|_{L_2(\Omega)} + \Big|\sum_{D\in\mathcal{P}} \int_D \nabla \times \boldsymbol{u}\, dx\Big| \Big)$$

provided

$$\pi_{\sigma,1}[\boldsymbol{u}]_\sigma = \mathbf{0}$$

Application to Nonconforming Finite Elements

Korn's inequality

$$|\boldsymbol{u}|_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|_{L_2(\Omega)} + \|\boldsymbol{u}\|_{L_2(\Omega)}\Big)$$

Korn's first inequality

$$|\boldsymbol{u}|_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|_{L_2(\Omega)} + \|\pi_{\Gamma,1}\boldsymbol{u}\|_{L_2(\Gamma)}\Big)$$

Korn's second inequality

$$|\boldsymbol{u}|_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|_{L_2(\Omega)} + \Big|\sum_{D\in\mathcal{P}}\int_D \nabla\times\boldsymbol{u}\,dx\Big|\Big)$$

or equivalently

$$\int_\sigma [\boldsymbol{u}]_\sigma \cdot \boldsymbol{l}\,ds = 0 \quad \forall\,\sigma \in S(\mathcal{P},\Omega),\ \boldsymbol{l} \in [P_1(\sigma)]^d$$

Application to Nonconforming Finite Elements

Korn's inequality

$$|\boldsymbol{u}|_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|_{L_2(\Omega)} + \|\boldsymbol{u}\|_{L_2(\Omega)}\Big)$$

Korn's first inequality

$$|\boldsymbol{u}|_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|_{L_2(\Omega)} + \|\pi_{\Gamma,1}\boldsymbol{u}\|_{L_2(\Gamma)}\Big)$$

Korn's second inequality

$$|\boldsymbol{u}|_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|_{L_2(\Omega)} + \Big|\sum_{D\in\mathcal{P}}\int_D \nabla\times\boldsymbol{u}\,dx\Big|\Big)$$

classical simplicial nonconforming finite elements of degree greater than or equal to 2 (Fortin-Soulie elements)

2D - Falk 1991

Application to Nonconforming Finite Elements

Korn's inequality

$$|\boldsymbol{u}|_{H^1(\Omega,\mathcal{P})} \le C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|_{L_2(\Omega)} + \|\boldsymbol{u}\|_{L_2(\Omega)}\Big)$$

Korn's first inequality

$$|\boldsymbol{u}|_{H^1(\Omega,\mathcal{P})} \le C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|_{L_2(\Omega)} + \|\pi_{\Gamma,1}\boldsymbol{u}\|_{L_2(\Gamma)}\Big)$$

Korn's second inequality

$$|\boldsymbol{u}|_{H^1(\Omega,\mathcal{P})} \le C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|_{L_2(\Omega)} + \Big|\sum_{D\in\mathcal{P}}\int_D \nabla\times\boldsymbol{u}\,dx\Big|\Big)$$

classical quadrilateral nonconforming finite elements such as Wilson's rectangle or brick (requires a slight modification of the arguments)

M. Wang 1994, Z. Zhang 1997, X. Xu 2000

Application to Mortar Methods

Korn's inequality

$$|\boldsymbol{u}|_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|_{L_2(\Omega)} + \|\boldsymbol{u}\|_{L_2(\Omega)}\Big)$$

Korn's first inequality

$$|\boldsymbol{u}|_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|_{L_2(\Omega)} + \|\pi_{\Gamma,1}\boldsymbol{u}\|_{L_2(\Gamma)}\Big)$$

Korn's second inequality

$$|\boldsymbol{u}|_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|_{L_2(\Omega)} + \Big|\sum_{D\in\mathcal{P}}\int_D \nabla \times \boldsymbol{u}\,dx\Big|\Big)$$

These inequalities can also be applied to mortar element functions (after a slight modification of the arguments).

Application to Discontinuous Galerkin Methods

Korn's inequality

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \le C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \|\boldsymbol{u}\|^2_{L_2(\Omega)} + \mathfrak{J}\Big)$$

Korn's first inequality

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \le C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \|\pi_{\Gamma,1}\boldsymbol{u}\|^2_{L_2(\Gamma)} + \mathfrak{J}\Big)$$

Korn's second inequality

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \le C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \Big|\sum_{D\in\mathcal{P}}\int_D \nabla\times\boldsymbol{u}\,dx\Big|^2 + \mathfrak{J}\Big)$$

$$\mathfrak{J} = \sum_{\sigma\in S(\mathcal{P},\Omega)} (\,\text{diam}\,\sigma\,)^{-1} \big\|\pi_{\sigma,1}[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)}$$

Application to Discontinuous Galerkin Methods

Korn's inequality

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \|\boldsymbol{u}\|^2_{L_2(\Omega)} + \mathfrak{J}\Big)$$

Korn's first inequality

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \|\boldsymbol{u}\|^2_{L_2(\Gamma)} + \mathfrak{J}\Big)$$

Korn's second inequality

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \Big|\sum_{D\in\mathcal{P}}\int_D \nabla\times\boldsymbol{u}\,dx\Big|^2 + \mathfrak{J}\Big)$$

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Application to Discontinuous Galerkin Methods

Korn's inequality

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Duarte, Carmo and Rochinha 2000

Strategy for "Discontinuation"

Strategy for Poincaré-Friedrichs Inequalities

Step 1 Find the correct form of the discontinuous Poincaré-Friedrichs inequality

$$\|u\|^2_{L_2(\Omega)} \leq C\left(|u|^2_{H^1(\Omega,\mathcal{P})} + \left|\int_\Omega u\,dx\right|^2 + ?\right)$$

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Determine a piecewise polynomial space V by setting the dominant term on the right-hand side to be zero.

Strategy for Poincaré-Friedrichs Inequalities

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$$\|u\|_{L_2(\Omega)}^2 \leq C\Big(|u|_{H^1(\Omega,\mathcal{P})}^2 + \Big|\int_\Omega u\,dx\Big|^2 + ?\Big)$$

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$V =$ the space of piecewise constant functions

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For $v \in V$, since $\pi_{\sigma,0}[v]_\sigma = [v]_\sigma$, we expect

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because $? = 0$ implies that $v = c$ is a global constant, in which case we have the trivial inequality

$$\|c\|^2_{L_2(\Omega)} \leq C\left|\int_\Omega c\,dx\right|^2$$

Strategy for Poincaré-Friedrichs Inequalities

This heuristic argument shows that the correct form of the discontinuous Poincaré-Friedrichs inequality is

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- First derive the inequality for functions in V (i.e., piecewise constant functions).
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Construction of $E : V \longrightarrow H^1(\Omega)$

Let $W \subset H^1(\Omega)$ be the P_1 finite element space associated with $\mathcal{T}$.

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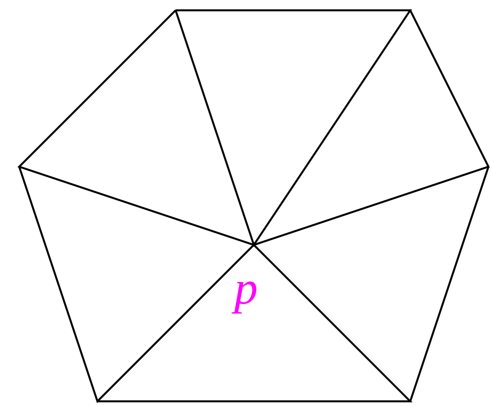

$E : V \longrightarrow W$ is defined by

$$(Ev)(p) = \frac{1}{|S_p|} \sum_{T \in S_p} v_{_T}(p)$$

$$S_p = \{T \in \mathcal{T} : \, p \in \partial T\} \qquad v_{_T} = v\Big|_T$$

Construction of $E : V \longrightarrow H^1(\Omega)$

Let $W \subset H^1(\Omega)$ be the P_1 finite element space associated with $\mathcal{T}$.

Basic Discrete Estimate

$$\sum_{p\in\mathcal{V}(\mathcal{T})} \left|(v - Ev)(p)\right|^2 \leq C \sum_{\sigma\in S(\mathcal{T},\Omega)} \sum_{p\in\mathcal{V}(\sigma)} \left|[v]_\sigma(p)\right|^2$$

$\mathcal{V}(\mathcal{T}) =$ the set of the vertices of $\mathcal{T}$

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Let $W \subset H^1(\Omega)$ be the P_1 finite element space associated with $\mathcal{T}$.

L_2 Estimate

$$\|v - Ev\|^2_{L_2(\Omega)} \leq C \sum_{\sigma \in S(\mathcal{T},\Omega)} (\,\text{diam}\,\sigma)\,\big\|\pi_{\sigma,0}[v]_\sigma\big\|^2_{L_2(\sigma)}$$

H^1 Estimate

$$|v - Ev|^2_{H^1(\Omega)} \leq C \sum_{\sigma \in S(\mathcal{T},\Omega)} (\,\text{diam}\,\sigma)^{-1}\,\big\|\pi_{\sigma,0}[v]_\sigma\big\|^2_{L_2(\sigma)}$$

Strategy for Poincaré-Friedrichs Inequalities

- First derive the inequality for functions in V (i.e., piecewise constant functions).
  - construct a linear map $E : V \longrightarrow H^1(\Omega)$
  - combine the classical Poincaré-Friedrichs inequality for Ev with the estimates for $v - Ev$ to obtain

$$\|v\|^2_{L_2(\Omega)} \leq C\Big(\Big|\int_\Omega v\,dx\Big|^2 + \sum_{\sigma\in S(\mathcal{P},\Omega)} (\,\text{diam}\,\sigma)^{-1}\|\pi_{\sigma,0}[v]_\sigma\|^2_{L_2(\sigma)}\Big)$$

for all $v \in V$

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- First derive the inequality for functions in V (i.e., piecewise constant functions).
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$$(\Pi u)\Big|_T = \frac{1}{|T|}\int_T u\,dx$$

  - derive the Poincaré-Friedrichs inequality for $H^1(\Omega, \mathcal{T})$ by combining the inequality for V and standard interpolation estimates for $u - \Pi u$

Strategy for Poincaré-Friedrichs Inequalities

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$$\|u\|^2_{L_2(\Omega)} \leq C\Big(|u|^2_{H^1(\Omega,\mathcal{T})} + \Big|\int_\Omega u\,dx\Big|^2 + \sum_{\sigma\in S(\mathcal{T},\Omega)} (\,\text{diam}\,\sigma)^{-1}\,\|\pi_{\sigma,0}[u]_\sigma\|^2_{L_2(\sigma)}\Big)$$

C depends only on the shape regularity of $\mathcal{T}$.

Strategy for Poincaré-Friedrichs Inequalities

More precisely, we have

$$\|u\|^2_{L_2(\Omega)} \leq \kappa(\theta_{\mathcal{T}})\Big(|u|^2_{H^1(\Omega,\mathcal{T})} + \Big|\int_\Omega u\,dx\Big|^2 + \sum_{\sigma\in S(\mathcal{T},\Omega)} (\,\text{diam}\,\sigma)^{-1}\,\|\pi_{\sigma,0}[u]_\sigma\|^2_{L_2(\sigma)}\Big)$$

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$\kappa : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a continuous function.

Strategy for Poincaré-Friedrichs Inequalities

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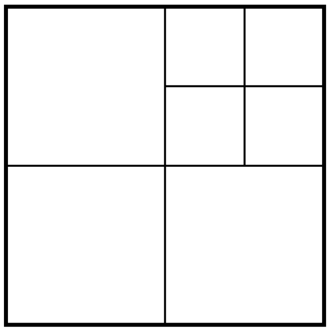
From Triangulation to Partition (2D)



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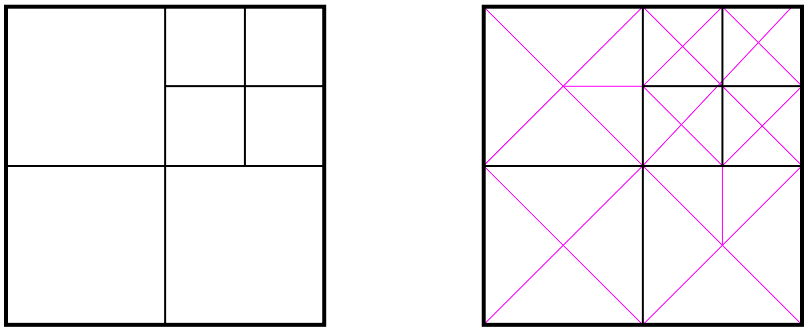

$\mathfrak{T}_{\mathcal{P}} = \{\mathcal{T} : \mathcal{T}$ is a triangulation of Ω by triangles and each member of $S(\mathcal{P}, \Omega)$ is also an edge of $\mathcal{T}\}$

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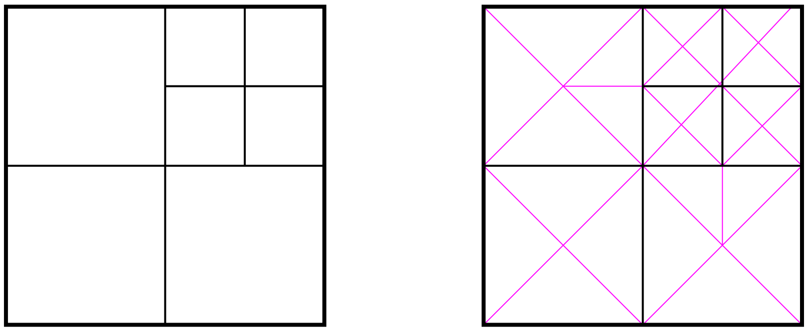

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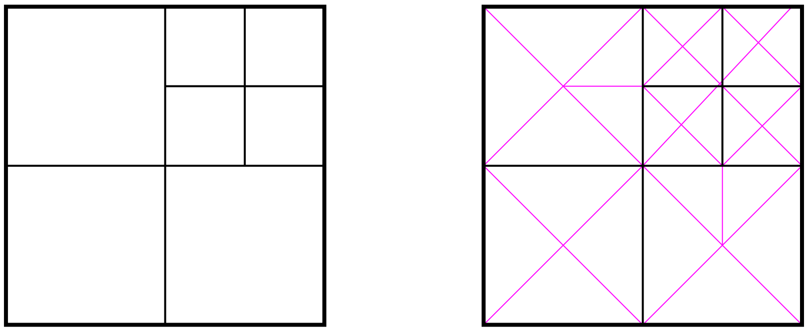

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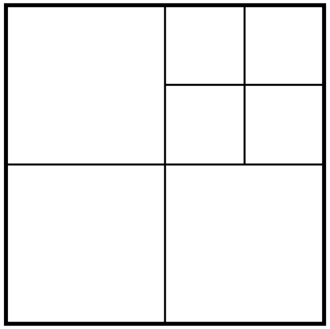

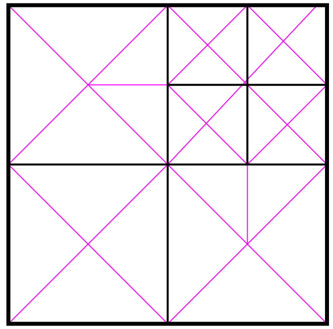

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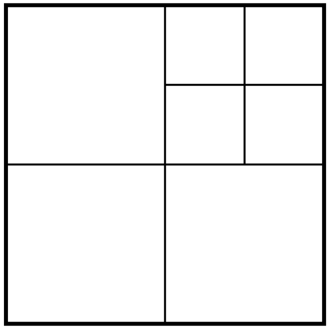 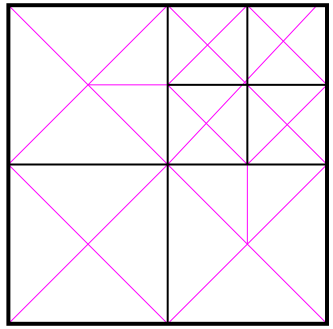

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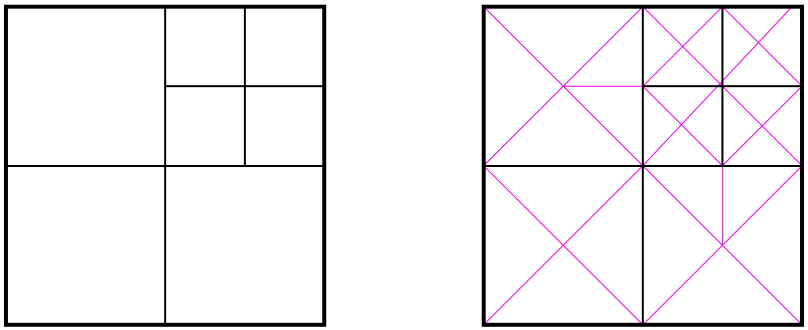

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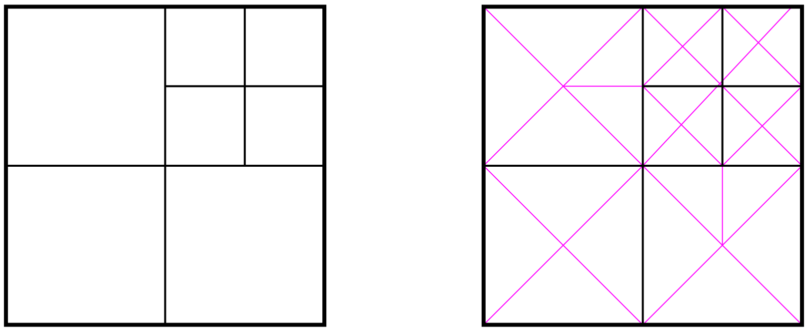

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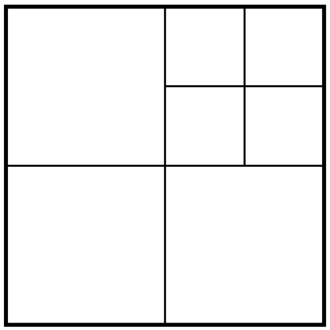

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Strategy for Poincaré-Friedrichs Inequalities

Step 1 Use a heuristic argument to determine the correct form of the inequality.

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This strategy also works for Korn's inequalities.

Strategy for Korn's Inequalities

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Step 1   Find the correct form of the discontinuous Korn's inequality

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$$\boldsymbol{x} = \begin{cases} (x_1, x_2)^t & d = 2 \\ (x_1, x_2, x_3)^t & d = 3 \end{cases}$$

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because $? = 0$ implies that $\boldsymbol{v}$ is a global rigid motion, in which case we have the trivial inequality

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\begin{aligned}
|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \le C\Big(&\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \|\boldsymbol{u}\|^2_{L_2(\Omega)} \\
&+ \sum_{\sigma\in S(\mathcal{P},\Omega)} \big(\operatorname{diam}\sigma\big)^{-1} \big\|\pi_{\sigma,1}[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)}\Big)
\end{aligned}
$$

Strategy for Korn's Inequalities

This heuristic argument shows that the correct form of the discontinuous Korn's inequality is

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{P})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \|\boldsymbol{u}\|^2_{L_2(\Omega)} + \sum_{\sigma\in S(\mathcal{P},\Omega)} (\,\text{diam}\,\sigma)^{-1}\big\|\pi_{\sigma,1}[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)}\Big)$$

Step 2 Derive the inequality when $\mathcal{P}$ is a simplicial triangulation $\mathcal{T}$.

Strategy for Korn's Inequalities

- First derive the inequality for vector fields in V (i.e., piecewise rigid motions).

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  - construct a linear map $E : V \longrightarrow [H^1(\Omega)]^d$
  - combine the classical Korn's inequality for $E\boldsymbol{v}$ with the estimates for $\boldsymbol{v} - E\boldsymbol{v}$ to obtain

$$|\boldsymbol{v}|^2_{H^1(\Omega,\mathcal{T})} \le C\Big(\|\boldsymbol{v}\|^2_{L_2(\Omega)} + \sum_{\sigma\in S(\mathcal{T},\Omega)} (\operatorname{diam}\sigma)^{-1} \big\|\pi_{\sigma,1}[\boldsymbol{v}]_\sigma\big\|^2_{L_2(\sigma)}\Big)$$

for all $\boldsymbol{v} \in V$

Strategy for Korn's Inequalities

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$$\Big| \int_T (\boldsymbol{u} - \Pi \boldsymbol{u})\, dx \Big| = 0 \quad \forall\, T \in \mathcal{T}$$

$$\Big| \int_T \nabla \times (\boldsymbol{u} - \Pi \boldsymbol{u})\, dx \Big| = 0 \quad \forall\, T \in \mathcal{T}$$

Strategy for Korn's Inequalities

- First derive the inequality for vector fields in V (i.e., piecewise rigid motions).
- Then derive the inequality for vector fields in $[H^1(\Omega, \mathcal{T})]^d$.
  - construct an operator $\Pi : [H^1(\Omega, \mathcal{T})]^d \longrightarrow V$

$$|\boldsymbol{u} - \Pi\boldsymbol{u}|_{H^1(T)} \leq C_T \|\boldsymbol{\epsilon}(\boldsymbol{u})\|_{L_2(T)}$$

where C_T depends only on the shape of T

Strategy for Korn's Inequalities

- First derive the inequality for vector fields in V (i.e., piecewise rigid motions).
- Then derive the inequality for vector fields in $[H^1(\Omega, \mathcal{T})]^d$.
  - construct an operator $\Pi : [H^1(\Omega, \mathcal{T})]^d \longrightarrow V$
  - derive Korn's inequality for $[H^1(\Omega, \mathcal{T})]^d$ by combining the inequality for V and the estimates for $\boldsymbol{u} - \Pi\boldsymbol{u}$

Strategy for Korn's Inequalities

- First derive the inequality for vector fields in V (i.e., piecewise rigid motions).
- Then derive the inequality for vector fields in $[H^1(\Omega,\mathcal{T})]^d$.

$$|\boldsymbol{u}|^2_{H^1(\Omega,\mathcal{T})} \leq C\Big(\|\boldsymbol{\epsilon}_{\mathcal{P}}(\boldsymbol{u})\|^2_{L_2(\Omega)} + \|\boldsymbol{u}\|^2_{L_2(\Omega)} + \sum_{\sigma\in S(\mathcal{T},\Omega)} (\operatorname{diam}\sigma)^{-1}\big\|\pi_{\sigma,1}[\boldsymbol{u}]_\sigma\big\|^2_{L_2(\sigma)}\Big)$$

C depends only on the shape regularity of $\mathcal{T}$.

Strategy for Korn's Inequalities

Step 1 Use a heuristic argument to determine the correct form of the inequality.

Step 2 Derive the inequality when $\mathcal{P}$ is a simplicial triangulation $\mathcal{T}$.

Step 3 Derive the inequality for a partition $\mathcal{P}$.

Summary

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- There are discontinuous versions of classical inequalities that can be applied to classical nonconforming finite element methods, mortar methods and discontinuous Galerkin methods.

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- Derivations of these inequalities involve only classical inequalities, discrete estimates and standard interpolation estimates.

- The "discontinuous versions" of these classical inequalities can be obtained systematically.

A Discrete Sobolev Inequality

A Discrete Sobolev Inequality

Ω is a bounded polygonal domain in $\mathbb{R}^2$.

A Discrete Sobolev Inequality

$\mathcal{T}_h$ is a family of quasi-uniform simplicial or quadrilateral triangulations of Ω.

A Discrete Sobolev Inequality

$V_h = \{v \in L_2(\Omega) : v\big|_D \in P_k(D) \; \forall \, D \in \mathcal{T}_h\}$ if $\mathcal{T}_h$ is a simplicial triangulation.

$V_h = \{v \in L_2(\Omega) : v\big|_D \in Q_k(D) \; \forall \, D \in \mathcal{T}_h\}$ if $\mathcal{T}_h$ is a quadrilateral triangulation.

A Discrete Sobolev Inequality

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$V_h = \{v \in L_2(\Omega) : v\big|_D \in Q_k(D) \; \forall \, D \in \mathcal{T}_h\}$ if $\mathcal{T}_h$ is a quadrilateral triangulation.

$$\|v\|^2_{L_\infty(\Omega)} \le C(1 + |\ln h|)\Big(\sum_{D \in \mathcal{T}_h} \|v\|^2_{H^1(D)} + \sum_{\sigma \in S(\mathcal{T}_h,\Omega)} (\,\text{diam}\,\sigma)^{-1} \|\pi_{\sigma,0}[v]_\sigma\|^2_{L_2(\sigma)}\Big) \qquad \forall\, v \in V_h$$

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$$\|v\|^2_{L_\infty(\Omega)} \leq C(1 + |\ln h|)\Big(\sum_{D \in \mathcal{T}_h} \|v\|^2_{H^1(D)} + \sum_{\sigma \in S(\mathcal{T}_h, \Omega)} (\,\text{diam}\, \sigma)^{-1} \|\pi_{\sigma,0}[v]_\sigma\|^2_{L_2(\sigma)} \Big) \qquad \forall \, v \in V_h$$

Discrete Sobolev and Poincaré inequalities for piecewise polynomial functions

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A Poincaré-Friedrichs Inequality for Piecewise H^2 Functions

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$|v|^2_{H^2(\Omega,\mathcal{P})} = \sum_{D\in\mathcal{P}} |v|^2_{H^2(D)}$

$$\begin{aligned}
\|v\|^2_{L_2(\Omega)} + |v|^2_{H^1(\Omega,\mathcal{P})} \le C\Big(&|v|^2_{H^1(\Omega,\mathcal{P})} + \|\pi_{\partial\Omega,1} v\|^2_{L_2(\partial\Omega)} \\
&+ \sum_{\sigma\in S(\mathcal{P},\Omega)} |\text{diam}\,\sigma|^{-3} \|\pi_{\sigma,1}[v]_\sigma\|^2_{L_2(\sigma)} \\
&+ \sum_{\sigma\in S(\mathcal{P},\Omega)} |\text{diam}\,\sigma|^{-1} \|\pi_{\sigma,0}[\partial v/\partial n]_\sigma\|^2_{L_2(\sigma)}\Big)
\end{aligned}$$

A Poincaré-Friedrichs Inequality for Piecewise H^2 Functions

$$|v|^2_{H^2(\Omega,\mathcal{P})} = \sum_{D\in\mathcal{P}} |v|^2_{H^2(D)}$$

$$\|v\|^2_{L_2(\Omega)} + |v|^2_{H^1(\Omega,\mathcal{P})} \leq C\Big(|v|^2_{H^1(\Omega,\mathcal{P})} + \|\pi_{\partial\Omega,1} v\|^2_{L_2(\partial\Omega)} + \sum_{\sigma\in S(\mathcal{P},\Omega)} |\text{diam}\,\sigma|^{-3}\|\pi_{\sigma,1}[v]_\sigma\|^2_{L_2(\sigma)} + \sum_{\sigma\in S(\mathcal{P},\Omega)} |\text{diam}\,\sigma|^{-1}\|\pi_{\sigma,0}[\partial v/\partial n]_\sigma\|^2_{L_2(\sigma)}\Big)$$

Poincaré-Friedrichs inequalities for piecewise H^2 functions

B., Wang and Zhao

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