



Ben-Gurion University of the Negev - ISRAEL  
Mechanical Engineering Dept.  
On Sabbatical at Div. of Applied Math. Brown Univ., USA

**Computing Singular Characteristics of the  
Elliptic 2-D and 3-D Problems Using p-FEM**

by

Zohar Yosibash  
[www.bgu.ac.il/~zohary](http://www.bgu.ac.il/~zohary)

---

Analytical & Numerical Treatment of Singularities in PDEs  
Oberwolfach - Nov. 2002

Anal. & Num. Treatment of Singularities in PDEs - 11-2002



**Many Thanks to the organizers  
for the invitation to such an interesting workshop!**

Anal. & Num. Treatment of Singularities in PDEs - 11-2002

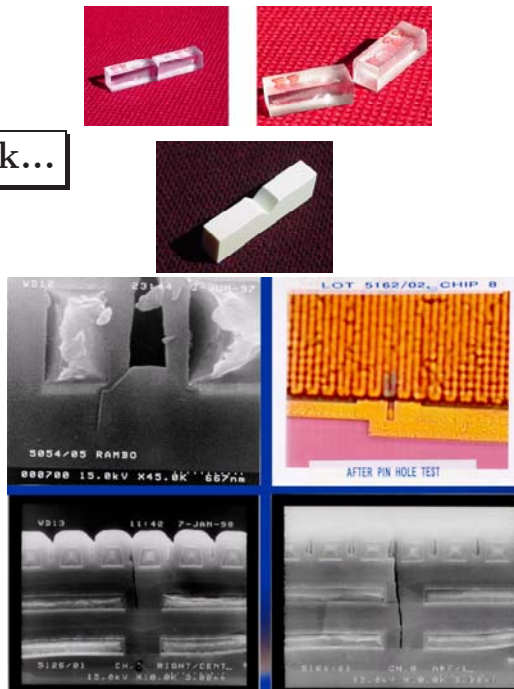
## Outline

- ☞ Some things in life simply fail... and sometimes it's really painful! Examples.
- ☞ 2-D linear elastic idealization - singular **points** :
  - Computation of eigen-pairs.
  - Extraction of the generalized flux/stress intensity factors.
- ☞ 3-D domains - singular **edges**, **vertices** and **vertex-edge** singularities:
  - Computation of **edge** eigen-pairs.
  - Computation of **vertex** eigen-pairs.
- ☞ Extraction of the **edge** generalized flux intensity functions.
- ☞ Some words on the non-homogeneous case: Thermo-elasticity.
- ☞ Summary.

Anal. & Num. Treatment of Singularities in PDEs - 11-2002

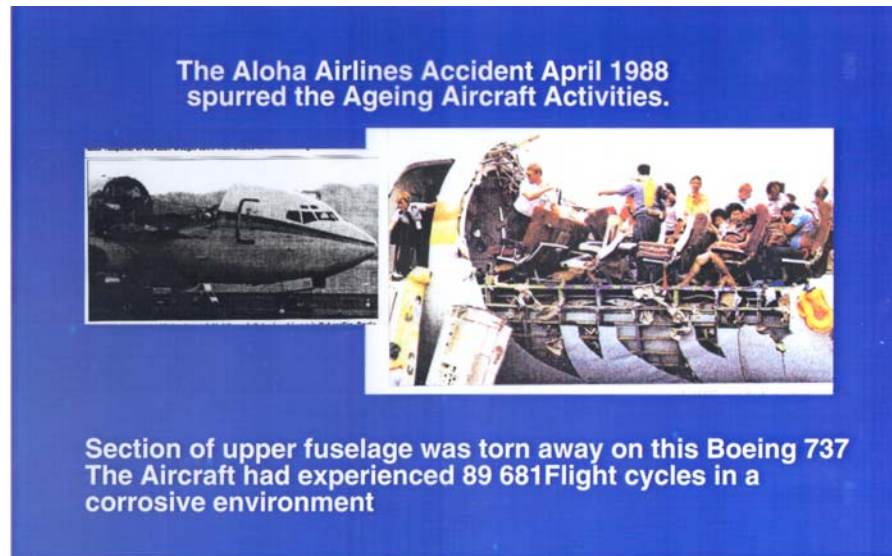
## Small things break...

- ☞ Composites, electronic devices and ceramic components may develop mechanical failures - of course at singular points:



Anal. & Num. Treatment of Singularities in PDEs - 11-2002

## And airplanes too...



Anal. & Num. Treatment of Singularities in PDEs - 11-2002

## What is it all about?

Can one provide a tool for predicting the onset of failures?

- ☞ For the special case of *cracks in isotropic* materials, failure initiation and propagation is successfully correlated to one parameter - stress intensity factor - associated with the *2-D linear elastic* solution in **the vicinity of the crack tip**.
- ☞ Is the Linear Elastic Fracture Mechanics theorem related to reality? Is it useful for engineering applications? This is another story now....

Anal. & Num. Treatment of Singularities in PDEs - 11-2002

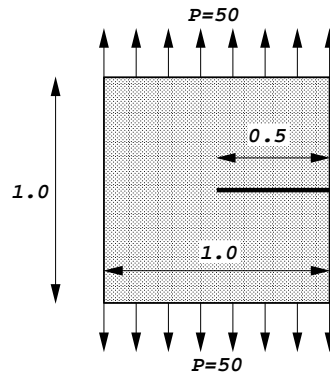
**Sometimes LEFM is good enough...**

Einstein's observation<sup>a</sup>:

Inasmuch as the mathematical theorems are related to reality, they are not sure; inasmuch as they are sure, they are not related to reality.

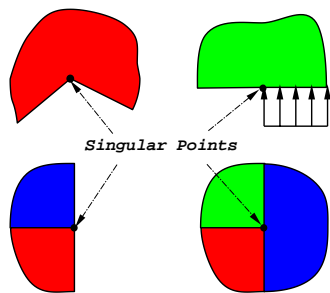
<sup>a</sup>Murphy's Laws - Book three, by Arthur Bloch (1987)

We looked at the following classical model problem and solved it twice:  
 Using linear theory of elasticity.  
 Using large-strains, large-deformations, elasto-plastic analysis ( $\sigma_y = 400$ ).



**The 2-D asymptotic expansion**

Crack tips in isotropic materials are only a minor subset of a very large family of singular points.



☞ In the neighborhood of a 2-D singular point,

$$\mathbf{u} = \sum_{i=1}^I \sum_{\ell=0}^L \sum_{m=0}^M A_{ilm} \cdot r^{\alpha_i+m} \cdot \mathbf{s}_{ilm}(\theta) \cdot \ln^\ell(r) + \mathbf{u}_{reg}$$

## Notation

- ☞  $\alpha_i$  are the eigenvalues,  $s_i(\theta)$  are the eigen-functions (together are the eigen-pairs).
- ☞  $A_i$  are generalized stress intensity factors GSIFs.
- ☞ For general anisotropic multi-material interfaces **neither the eigen-pairs nor the GSIFs are known.**

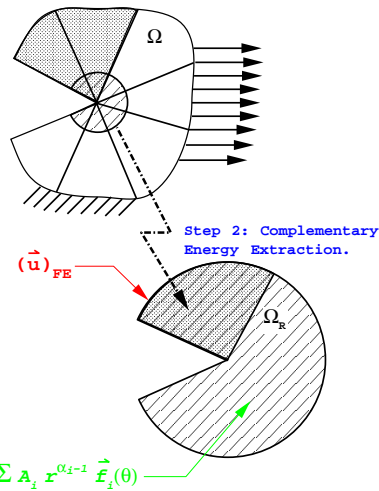
## We wish to

Develop a **reliable, accurate and efficient** numerical tool for determining the solution (i.e. eigen-pairs and GSIFs) in the vicinity of singular points for **general 2-D (3-D) domains.**

## Overall Numerical Procedure

Step 0: Compute eigen-pairs -  $\alpha_i$  &  $f_i(\theta)$

Step 1: FE solution.



Step 0: Eigen-pairs computation by the modified Steklov method.

Step 1: F.E. solution over the whole domain.

Step 2: F.E. solution in conjunction with eigen-pairs are used to compute GSIFs.



## Computation of 2-D Eigen-pairs: Survey

- ☞ Leguillon & Sanchez-Palencia (87): Matrix & the determinant methods (h-FEM). Reducing the problem to a 1-D quadratic eigen-problem.
- ☞ Papadakis & Babuška (88): Matrix method, followed by the shooting-method (1-D quadratic eigen-problem) - PRECISE.
- ☞ Barsoum (88): Iterative method based on h-FEM for the first eigen-value.
- ☞ Yosibash & Szabó (93): Modified Steklov Method. (Stress Check).
- ☞ Babuška, von-Petersdorff & Andersson: Steklov Method for 3-D Laplacian.

Anal. & Num. Treatment of Singularities in PDEs - 11-2002



## Computation of Eigen-pairs: Cont. Survey

- ☞ Gu and Belytschko (94): Method based on stress function and interpolation of displacements (h-FEM).
- ☞ Costabel and Dauge (95): An accurate matrix method (splitting the differential operator).
- ☞ Pageau et al. (95): h-FEM and solution of a quadratic eigen-problem.
- ☞ Costabel, Dauge, Lafranche, Y. (98): A fast semi-analytic method.

Anal. & Num. Treatment of Singularities in PDEs - 11-2002

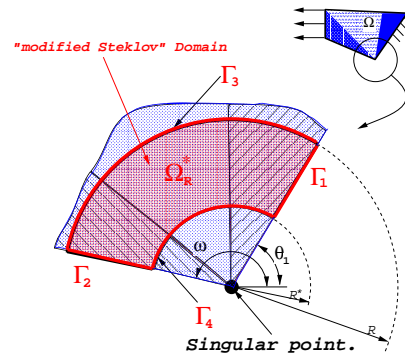
## Modified Steklov Method - Laplace eq.

- ☞ Explaining the modified Steklov method for eigen-pairs computation on the simplified Laplace equation.
- ☞ A *localized domain* in the vicinity of the singular point is considered.

☞ We wish to solve:

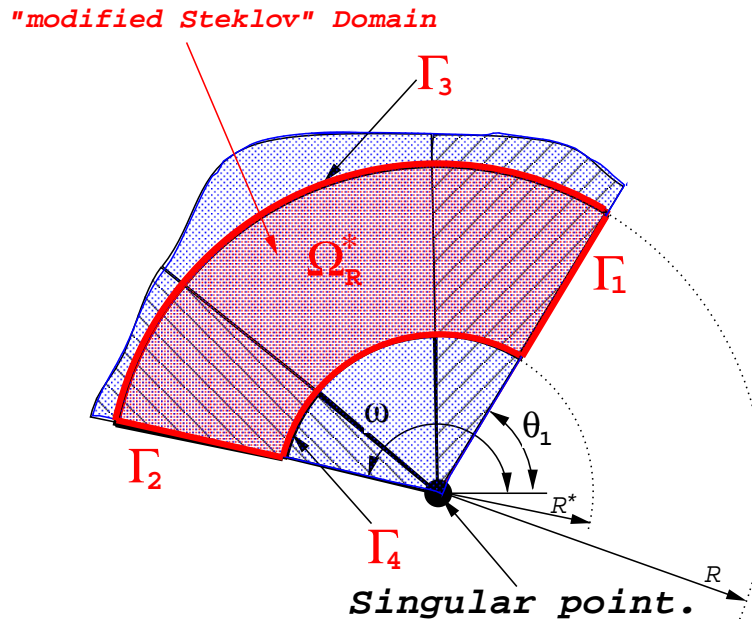
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } \Omega$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma_1, \Gamma_2$$



Anal. & Num. Treatment of Singularities in PDEs - 11-2002

## The “Modified Steklov Domain”



Anal. & Num. Treatment of Singularities in PDEs - 11-2002



### Modified Steklov Method - Laplace eq.

☞ In  $\Omega_R^*$ , a solution is of the form  $u = r^\alpha s(\theta)$  satisfies the PDE and BCs, so:

$$\frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial r} = \alpha r^{(\alpha-1)} s(\theta) \Rightarrow \frac{\partial u}{\partial r} = (\alpha/R)u \quad \text{on } \Gamma_3, \Gamma_4.$$

☞ To summarize, we have to solve the *Strong Steklov* problem:

$$\begin{aligned} \nabla^2 u &= 0 && \text{in } \Omega_R^* \\ \frac{\partial u}{\partial \theta} &= 0 && \text{on } \Gamma_1, \Gamma_2 \\ \frac{\partial u}{\partial r} &= (\alpha/R)u && \text{on } \Gamma_3, \Gamma_4. \end{aligned}$$



### Modified Steklov Method - Laplace eq.

#### The Weak Formulation

☞ The Strong Steklov problem is cast as a weak formulation, suitable for FE analysis:

Seek  $\alpha \in \mathcal{R}$ ,  $0 \neq u \in H^1(\Omega_R^*)$ , such that  $\forall v \in H^1(\Omega_R^*)$ ,

$$\mathcal{B}(u, v) = \alpha (\mathcal{M}_{R^*}(u, v) + \mathcal{M}_R(u, v))$$

$$\mathcal{B}(u, v) = \iint_{\Omega_R^*} \sum_{i,j=1}^2 \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} d\Omega$$

$$\mathcal{M}_R(u, v) = \int_{\theta} (u v)_{r=R} d\theta$$



### Modified Steklov Method - Elasticity

☞ For Navier-Lamme eqs. (linear elasticity):

$$\begin{aligned} \text{Seek } \alpha \in \mathcal{C}, \mathbf{0} \neq \mathbf{u} \in [H^1(\Omega_R^*)]^2, \text{ s. t. } \forall \mathbf{v} \in [H^1(\Omega_R^*)]^2 \\ \mathcal{B}(\mathbf{u}, \mathbf{v}) - (\mathcal{N}_R(\mathbf{u}, \mathbf{v}) + \mathcal{N}_{R^*}(\mathbf{u}, \mathbf{v})) \\ = \alpha (\mathcal{M}_R(\mathbf{u}, \mathbf{v}) + \mathcal{M}_{R^*}(\mathbf{u}, \mathbf{v})), \end{aligned}$$

$$\mathcal{M}_R(\mathbf{u}, \mathbf{v}) = \int_{\theta} \left[ \{v^{(x,y)}\}^T [A_1]^T [A_3] [E] [A_5] \{u^{(x,y)}\} \right]_{r=R} d\theta$$

$$\mathcal{N}_R(\mathbf{u}, \mathbf{v}) = \int_{\theta} \left[ \{v^{(x,y)}\}^T [A_1]^T [A_3] [E] [D(\theta)] \{u^{(x,y)}\} \right]_{r=R} d\theta .$$

Anal. & Num. Treatment of Singularities in PDEs - 11-2002



### Modified Steklov Method - Numerical Scheme

- ☞ Weak form = Saddle point problem.
- ☞ The weak form does not exclude the existence of negative eigenvalues. Functions of the form  $r^{-\alpha} f(\theta)$  belong in the space  $H^1(\Omega_R^*) \rightarrow$  both the positive and negative eigen-pairs are obtained.
- ☞ No singularities in  $\Omega_R^*$  hence p-FEM provides exponential convergence rate.

Anal. & Num. Treatment of Singularities in PDEs - 11-2002

### Modified Steklov Method - Numerical Scheme

- ☞ Matrix formulation of weak form: (Add coefficients of basis functions associated with  $\Gamma_3$  to those associated with  $\Gamma_4$ , denoting them by the vector  $\vec{u}_{RR^*}$ ):

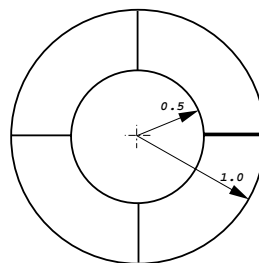
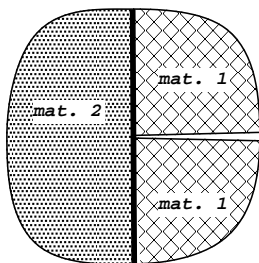
$$\begin{bmatrix} \mathbf{K}_{RR^*} & \mathbf{K}_{RR^*-I} \\ \mathbf{K}_{I-RR^*} & \mathbf{K}_I \end{bmatrix} \begin{Bmatrix} \vec{u}_{RR^*} \\ \vec{u}_I \end{Bmatrix} = \alpha \begin{bmatrix} \mathbf{M}_{RR^*} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \vec{u}_{RR^*} \\ \vec{u}_I \end{Bmatrix}.$$

- ☞  $\vec{u}_I$  is eliminated by static condensation, obtaining the “tiny” eigen-problem:

$$(\mathbf{K}_{RR^*} - \mathbf{K}_{RR^*-I} \cdot \mathbf{K}_I^{-1} \cdot \mathbf{K}_{I-RR^*}) \vec{u}_{RR^*} = \alpha \mathbf{M}_{RR^*} \vec{u}_{RR^*}$$

### Eigen-pairs: Crack Perpendicular to Interface

- ☞ Consider the elastic domain with a crack perpendicular to the interface.



$$\begin{aligned} E_1 &= 1.925925925, \\ \nu_1 &= 0.3 \\ E_2 &= 1.0, \quad \nu_2 = 0.35 \\ &\text{Plane Stress} \end{aligned}$$

### Eigen-pairs: Crack Perpendicular to Interface

☞ The exact first *double* eigen-value is 0.42944.

Relative error (%) in the first eigenvalue

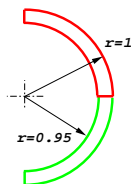
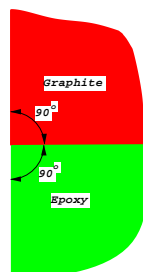
	<u>p=1</u>	<u>p=2</u>	<u>p=3</u>	<u>p=4</u>	<u>p=5</u>	<u>p=6</u>	<u>p=7</u>	<u>p=8</u>
$e_{\alpha_1}^{(1)}$	-1.7	-0.14	-0.15	-0.04	-0.0046	-0.002	**	**
$e_{\alpha_1}^{(2)}$	-0.76	-0.05	-0.009	0.01	-0.0056	**	**	**
dof.	20	46	72	106	148	198	256	322

\*\* Numerical result coincides with the analytical solution up to fifth digit.

### Modified Steklov Method - Numerical Example

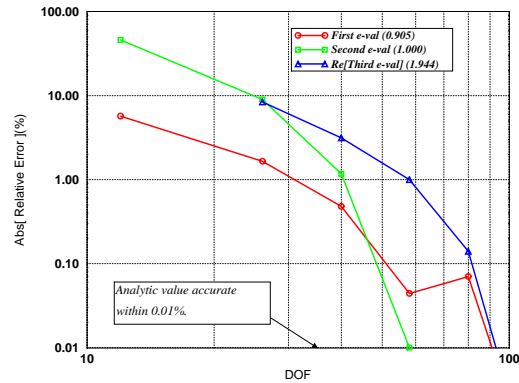
☞ Consider two orthotropic materials, graphite and adhesive (epoxy), bonded together, with plane strain condition assumed.

Material properties.



	Graphite	Adhesive
$E_{11} \times 10^6$ psi	20.	1.4
$E_{22}$	2.	1.4
$E_{33}$	2.	1.4
$\nu_{12}$	0.450	0.3
$\nu_{23}$	0.040	0.3
$\nu_{31}$	0.045	0.3
$G_{12} \times 10^6$ psi	1.1	0.7

## Modified Steklov Method - Numerical Example

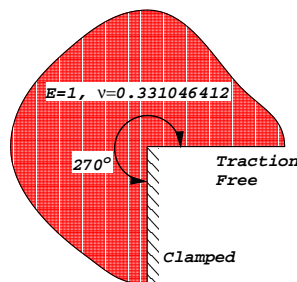


- At  $p$ -level = 6 the numerical approximation of the eigenvalues is as good as the reported exact eigenvalues.

Anal. & Num. Treatment of Singularities in PDEs - 11-2002

## Power-Logarithmic Stress Singularity

- For a specific combination of geometry and material properties power-logarithmic stress singularity,  $\mathcal{O}(r^{\alpha-1} \ln r)$ , may occur.



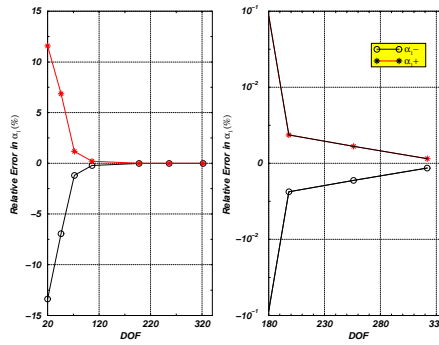
- Wedge configuration exciting power-logarithmic stress singularity:

$$\sigma = A_1 r^{0.34254974-1} \ln r s_1(\theta) + A_2 r^{0.34254974-1} s_2(\theta) + \text{h.o.t.}$$

Anal. & Num. Treatment of Singularities in PDEs - 11-2002

## Power-Logarithmic Stress Singularity

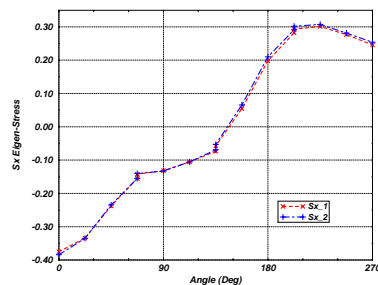
☞ The first two eigen-pairs obtained on a 4 element mesh using the modified Steklov method collapse into a single one:



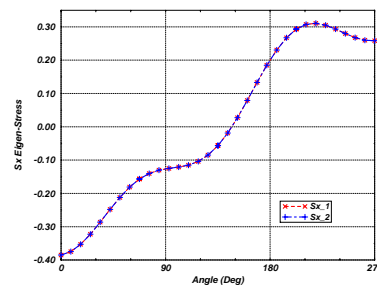
Anal. & Num. Treatment of Singularities in PDEs - 11-2002

## Power-Logarithmic Stress Singularity

☞ The first two eigen-stresses collapse into a single one. For example,  $\sigma_x$ :



$p = 3.$



$p = 8.$

Anal. & Num. Treatment of Singularities in PDEs - 11-2002



## Generalized Stress Intensity Factors

- ☞ Once eigen-pairs are computed one may proceed to the second task of computing the **GSIFs**.
- ☞ A p-version finite element solution of the overall domain of interest is obtained.
- ☞ The eigen-pairs + the finite element solution in the vicinity of the singular point + complementary energy principle  $\implies$  The GSIFs.

Anal. & Num. Treatment of Singularities in PDEs - 11-2002



## Generalized Stress Intensity Factors

- ☞ To use the Complementary Energy Principle, one needs to define the *Statically Admissible Space*:

$$\Sigma(\Omega_R) = \{ \boldsymbol{\sigma} \mid \mathbf{div} \boldsymbol{\sigma} = 0, (\boldsymbol{\sigma}, \boldsymbol{\sigma})_{L^2} < \infty \}$$

- ☞ The basis functions which span the finite dimensional statically admissible space  $\Sigma^N(\Omega_R)$  are constructed using the eigen-pairs:

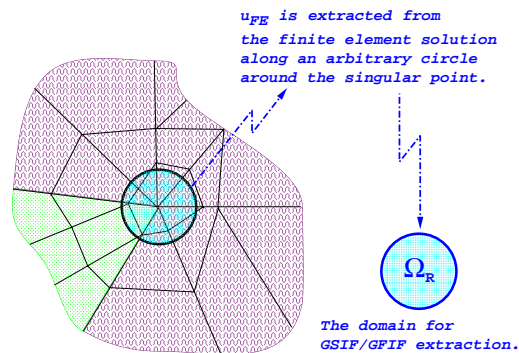
$$\boldsymbol{\sigma} = \sum_{n=1}^N A_n \alpha_n r^{\alpha_n - 1} [E][D] \mathbf{s}_n(\theta)$$

Anal. & Num. Treatment of Singularities in PDEs - 11-2002

## Generalized Stress Intensity Factors

Seek  $\boldsymbol{\sigma} \in \Sigma(\Omega_R)$  such that

$$\iint_{\Omega_R} \boldsymbol{\sigma}^T [E]^{-1} \boldsymbol{\sigma}_o d\Omega = \int_{\Gamma_R} \mathbf{u}_{FE}^T [A] \boldsymbol{\sigma}_o ds \quad \forall \boldsymbol{\sigma}_o \in \Sigma(\Omega_R).$$



Anal. & Num. Treatment of Singularities in PDEs - 11-2002

## Generalized Stress Intensity Factors

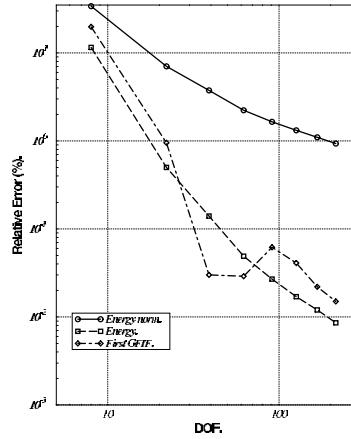
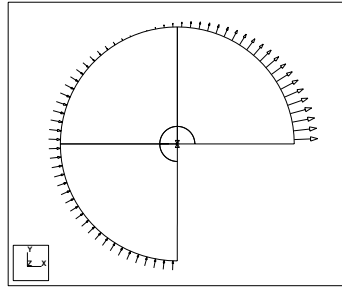
- ☞ LHS of the complementary weak form can be pre-computed from the eigen-pairs alone. The dimension of the resulting matrix is very small.
- ☞ RHS depends on the eigen-pairs and the FE displacements along a circular path.
- ☞ Solving the complementary weak formulation, one obtains directly the GSIFs.
- ☞ For the Laplace problem we were able to prove mathematically that the GFIFs converge at a rate which is faster than the energy norm. Numerical experiments show “*superconvergence*”.

Anal. & Num. Treatment of Singularities in PDEs - 11-2002

### GFIFs - Numerical Example: "Laplace"

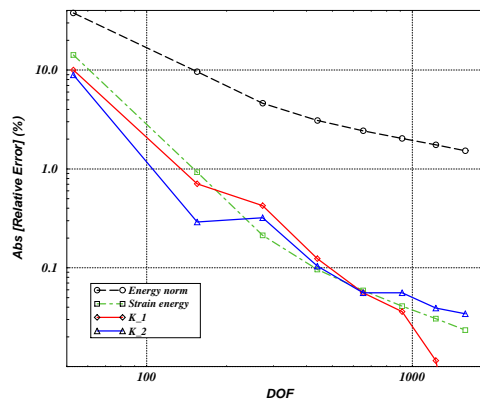
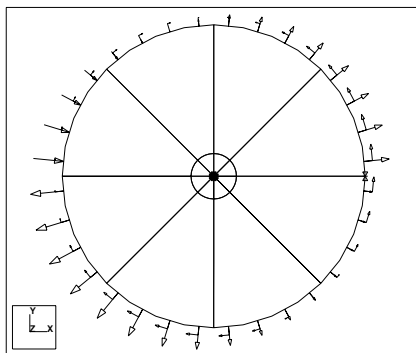
$$4 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{on unit circle sector}$$

On circular boundary Neumann b.c. corresponding to first eigen-pair is applied.



Anal. & Num. Treatment of Singularities in PDEs - 11-2002

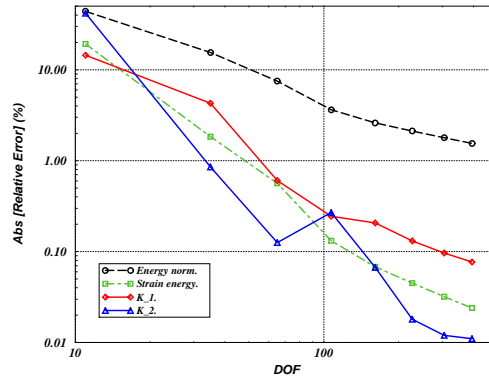
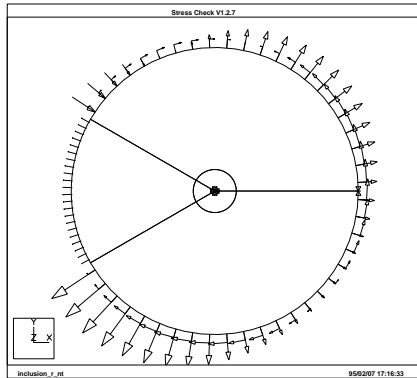
### GSIFs - Numerical Example: "Dissimilar" Crack



$E_{up} = 10$ ,  $\nu_{up} = 0.3$ ,  $E_{down} = 1$ ,  $\nu_{down} = 0.3$ . Mesh and loading for the crack between dissimilar materials, and convergence of GSIFs.

Anal. & Num. Treatment of Singularities in PDEs - 11-2002

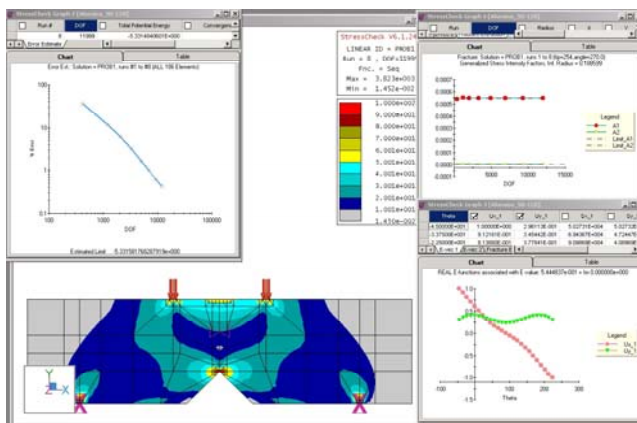
## GSIFs - Numerical Example: Inclusion Problem



Plane- $\varepsilon$ ,  $\nu_1 = \nu_2 = 0.3$  and  $E_1 = 10,000$ ,  $E_2 = 1$ .  $\alpha_1 = 0.512472160$  and  $\alpha_2 = 0.730975740$ . Using two terms in the series, and an integration path with  $R = 0.8$ ,

Anal. & Num. Treatment of Singularities in PDEs - 11-2002

## Implementation in a Commercial p-FEM code



A practical example - Computation of eigen-pairs and GSIFs for a four-point V-notched specimen.

Anal. & Num. Treatment of Singularities in PDEs - 11-2002



## What can we say on 2-D?

- ✎ Eigen-pairs and GSIFs can be computed accurately, reliably, and methods are robust.
- ✎ Methods implemented in a commercial FE code (Stress Check), and successfully used by researchers and engineers.

Bitton's postulate on state-of-the-art in science<sup>a</sup>:

If you understand it, it's obsolete.

✎ So, what about 3-D?

---

<sup>a</sup>Murphy's Law - Book three, by Arthur Bloch (1987)

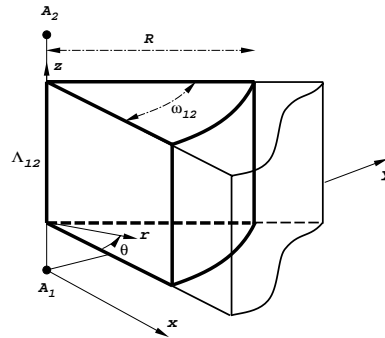


## Assumptions - Edge eigen-pairs

- ✎ Consider only straight edges, and that crack faces, if any, lay in a flat plane.
- ✎ Assumed that the edge of interest lays along the  $z$  Cartesian axis, with the geometry and material properties independent of it. Let us denote  $\mathbf{u} \stackrel{\text{def}}{=} (u_x, u_y, u_z)^T$ .

### Some Notations

- ☞ A cylindrical domain of radius  $r = R$  having the edge  $\Lambda_{ij}$  as its axis is considered.



- ☞ In the edge neighborhood:

$$\mathbf{u}(r, \theta, z) = \sum_{k=1}^K \sum_{\ell=0}^L a_{k\ell}(z) r^{\alpha_k} (\ln r)^\ell \mathbf{s}_{k\ell}(\theta) + \mathbf{u}_{reg}(r, \theta, z)$$

### Edge Eigen-pairs

- ☞  $\alpha_{k+1} \geq \alpha_k$  are  $z$ -independent and so are  $\vec{f}_{ks}(\theta)$  which are also analytic in  $\theta$ .
- ☞  $a_{ks}(z)$  are analytic in  $z$  called edge stress intensity functions: *ESIFs* (can become very large or zero as they approach one of the vertices).
- ☞ The vector functions  $\mathbf{u}_{reg}(r, \theta, z)$  are smooth.
- ☞ For  $L = 0$ :

$$\mathbf{u}(r, \theta, z) = \sum_{k=1}^K a_k(z) r^{\alpha_k} \mathbf{s}_k(\theta) + \mathbf{u}_{reg}(r, \theta, z).$$



### Edge Eigen-pairs - Modified Steklov form

- ☞ On the boundaries  $\theta = 0$  and  $\theta = \omega_{ij}$  homogeneous boundary conditions are assumed.
- ☞ Any  $\mathbf{u}$  of the form  $\mathbf{u} = A(z)\tilde{\mathbf{u}}(r, \theta)$  satisfies the equilibrium eqs. if  $\tilde{\mathbf{u}}(r, \theta) = r^\alpha \mathbf{s}(\theta)$ , then after cumbersome mathematical manipulations:

Anal. & Num. Treatment of Singularities in PDEs - 11-2002



### Edge Eigen-pairs (cont.)

$$\begin{aligned}
 & \int_{R^*}^R \int_0^{\omega_{12}} \left\{ ([A_r] \partial_r + [A_\theta] \frac{\partial}{\partial \theta}) \tilde{\mathbf{v}} \right\}^T [E] \left\{ ([A_r] \partial_r + [A_\theta] \frac{\partial}{\partial \theta}) \tilde{\mathbf{u}} \right\} r d\theta dr \\
 & \quad - \int_0^{\omega_{12}} \tilde{\mathbf{v}}^T [A_r]^T [E] [A_\theta] \partial_\theta \tilde{\mathbf{u}} \Big|_{r=R^*}^{r=R} d\theta \\
 & = \alpha \int_0^{\omega_{12}} \tilde{\mathbf{v}}^T [A_r]^T [E] [A_r] \tilde{\mathbf{u}} \Big|_{r=R^*}^{r=R} d\theta \\
 & + \frac{\int_0^\Delta A''(z) B(z) dz}{\int_0^\Delta A(z) B(z) dz} \int_{R^*}^R \int_0^{\omega_{12}} \tilde{\mathbf{v}}^T [A_z]^T [E] [A_z] \tilde{\mathbf{u}} r d\theta dr \\
 & + \frac{\int_0^\Delta A'(z) B(z) dz}{\int_0^\Delta A(z) B(z) dz} \int_0^{\omega_{12}} r \tilde{\mathbf{v}}^T [A_r]^T [E] [A_z] \tilde{\mathbf{u}} \Big|_{r=R^*}^{r=R} d\theta.
 \end{aligned}$$

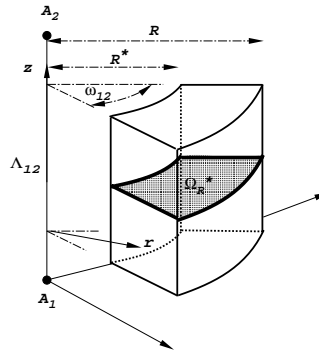
Anal. & Num. Treatment of Singularities in PDEs - 11-2002

## Modified Steklov Weak Formulation - Edge

☞ Eigen-pairs are independent of  $z$ , so we obtain:

$$\text{Seek } \alpha \in \mathcal{C}, \quad \mathbf{0} \neq \tilde{\mathbf{u}} \in [H^1(\tilde{\Omega}_R^*)]^3, \quad \text{s.t.}, \quad \forall \tilde{\mathbf{v}} \in [H^1(\tilde{\Omega}_R^*)]^3$$

$$\tilde{\mathcal{B}}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) - [\mathcal{N}_R(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) - \mathcal{N}_{R^*}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})] = \alpha [\mathcal{M}_R(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) - \mathcal{M}_{R^*}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})]$$



Anal. & Num. Treatment of Singularities in PDEs - 11-2002

## Mod. Steklov Weak Form - Edge (cont.)

$$\tilde{\mathcal{B}}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \stackrel{\text{def}}{=} \int_{R^*}^R \int_0^{\omega_{12}} \left\{ \left( [A_r] \partial_r + [A_\theta] \frac{\partial_\theta}{r} \right) \tilde{\mathbf{v}} \right\}^T \cdot [E] \left\{ \left( [A_r] \partial_r + [A_\theta] \frac{\partial_\theta}{r} \right) \tilde{\mathbf{u}} \right\} r d\theta dr$$

$$\mathcal{N}_R(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \stackrel{\text{def}}{=} \int_0^{\omega_{12}} \tilde{\mathbf{v}}^T [A_r]^T [E] [A_\theta] \partial_\theta \tilde{\mathbf{u}} \Big|_{r=R} d\theta$$

$$\mathcal{M}_R(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) \stackrel{\text{def}}{=} \int_0^{\omega_{12}} \tilde{\mathbf{v}}^T [A_r]^T [E] [A_r] \tilde{\mathbf{u}} \Big|_{r=R} d\theta$$

Anal. & Num. Treatment of Singularities in PDEs - 11-2002

### Mod. Steklov Weak Form - Edge (cont.)

- ☞ The test and trial functions have three components, but the domain over which the weak eigen-formulation is defined is two-dimensional, and excludes any singular points. Therefore the application of the p-version of the FEM is expected to be very efficient.
- ☞ FE implementation is very similar to 2-D.

### A Two Cross-Ply Anisotropic Laminate

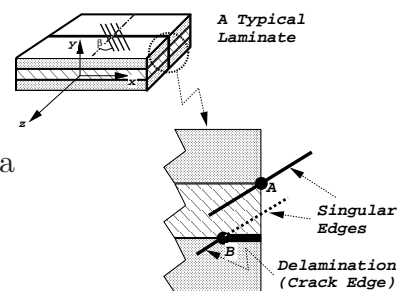
- ☞ Consider a graphite-epoxy laminate with  $\nu_{LT} = \nu_{Lz} = \nu_{Tz} = 0.21$ ,

$$G_{LT} = G_{Lz} = G_{Tz} = 0.586 \times 10^4 \text{MPa}$$

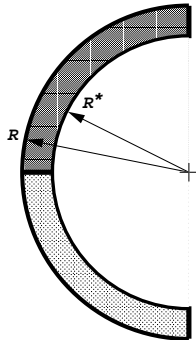
$$E_L = 1.38 \times 10^5 \text{MPa}$$

$$E_T = E_z = 1.45 \times 10^4 \text{MPa}$$

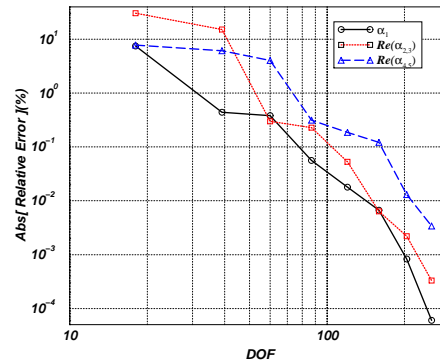
- ☞ For a  $\pm\beta = 45^\circ$  angle-ply composite the exact first five eigen-values are:  $\alpha_1 = 0.974424342$ ,  
 $\alpha_{2,3} = 1.88147184 \pm i0.23400497$ ,  
 $\alpha_{4,5} = 2.5115263 \pm i0.79281732$



## A Two Cross-Ply Anisotropic Laminate



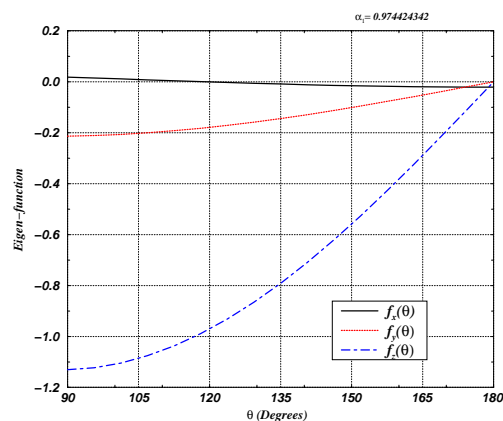
The FE mesh used for computations.



Convergence of approximated eigen-values.

Anal. & Num. Treatment of Singularities in PDEs - 11-2002

## A Two Cross-Ply Anisotropic Laminate



The 3-D eigen-function vector associated with  $\alpha_1$  obtained at  $p = 8$ .

Anal. & Num. Treatment of Singularities in PDEs - 11-2002

### Anisotropic Multi-Material Internal Interface

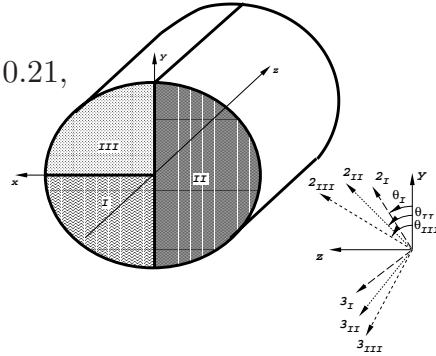
Consider a three material internal interface made of same fiber/resin composite:

$$\nu_{12} = 0.02205, \nu_{13} = 0.21, \nu_{23} = 0.21,$$

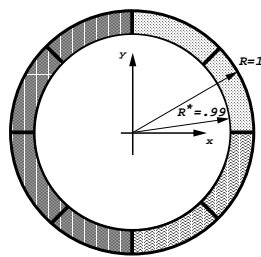
$$E_1 = E_3 = 0.105, E_2 = 1.0$$

$$G_{12} = G_{13} = G_{23} = 0.0425,$$

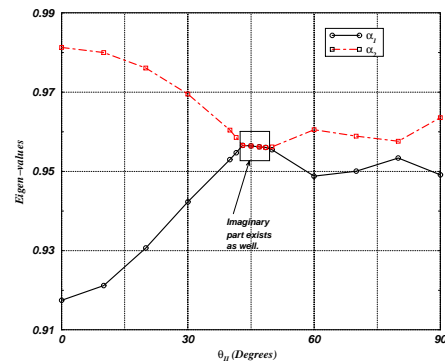
subscripts 1, 2, 3 refer to fiber, transverse and thickness directions.



### Anisotropic Multi-Material Internal Interface



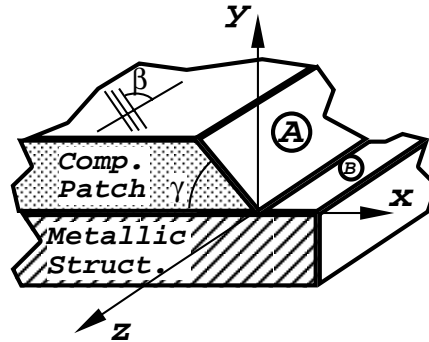
The FE mesh used for computations.



Variation of eigen-values as  $\theta_{II}$  changes,  $\theta_I = -\theta_{III} = -45^\circ$ .

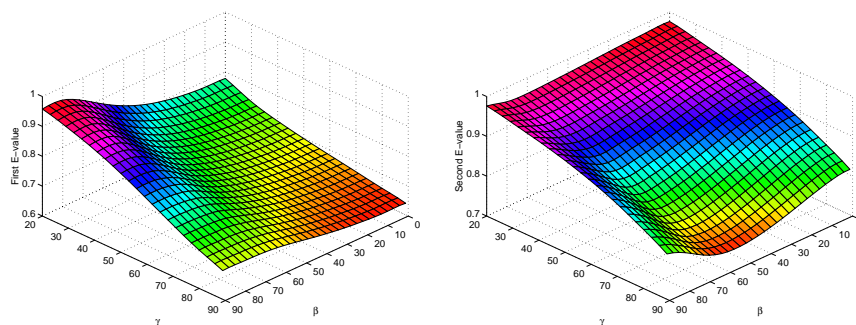
### Composite Patch Attached to a Metallic Struct.

- Consider laminated patch, terminating at an angle  $\gamma$ , attached to the metallic structure.
- The patch is of graphite-epoxy lamina, with fibers orientation rotated by an angle  $\beta$  about the  $y$ -axis.
- The metallic structure is AL7075-T6 with  $E = 7.17 \times 10^4 \text{MPa}$ , and  $\nu = 0.3$ .



Anal. & Num. Treatment of Singularities in PDEs - 11-2002

### Composite Patch Attached to a Metallic Struct.



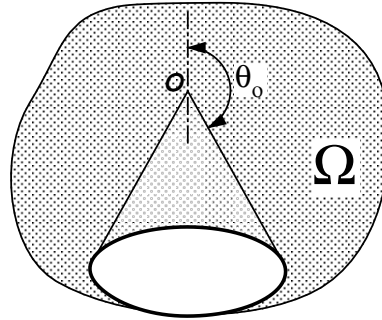
First two eigen-values as a function of the fiber orientation angle ( $\beta$ ), and terminating angle  $\gamma$ . FE mesh consists of six elements.

- These plots provide optimal combination of angles  $\beta$  and  $\gamma$  to produce the highest first two eigen-values.

Anal. & Num. Treatment of Singularities in PDEs - 11-2002

### Modified Steklov Formulation - Vertex

☞ Consider a 3-D domain  $\Omega$  having a rotationally symmetric conical vertex  $O$ .

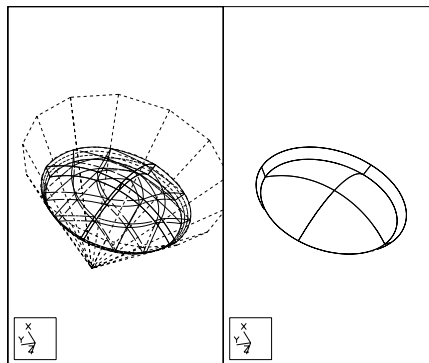


Locating a spherical coordinate system in  $O$ :

$$\begin{cases} u_x \\ u_y \\ u_z \end{cases} \stackrel{\text{def}}{=} \mathbf{u}(r, \theta, \phi) = \sum_{k=1}^K \sum_{\ell=0}^L a r^{\alpha_k} (\ln r)^\ell \mathbf{s}_{k\ell}(\theta, \phi) + \mathbf{u}_{reg}$$

### Modified Steklov Formulation - Vertex

☞ We consider the sub-domain  $\Omega^*$  bounded by the cone surface and two spheres centered at the conical point of radii  $R > R^*$ .





### Modified Steklov Formulation - Vertex

☞ Any eigen-pair on the artificial sphere of radius  $R$  satisfies

$$\frac{\partial \mathbf{u}}{\partial n} = \frac{\alpha}{R} \mathbf{u}.$$

☞ Therefore we obtain the weak Steklov eigen-problem:

$$\text{Seek } \alpha \in \mathcal{C}, \quad \mathbf{0} \neq \mathbf{u} \in [H^1(\Omega^*)]^3, \quad \text{such that,} \quad \forall \mathbf{v} \in [H^1(\Omega^*)]^3$$

$$\mathcal{B}(\mathbf{u}, \mathbf{v}) - [\mathcal{N}_R(\mathbf{u}, \mathbf{v}) + \mathcal{N}_{R^*}(\mathbf{u}, \mathbf{v})] = \alpha [\mathcal{M}_R(\mathbf{u}, \mathbf{v}) + \mathcal{M}_{R^*}(\mathbf{u}, \mathbf{v})]$$

Here  $\mathcal{B}(\mathbf{u}, \mathbf{v})$  is the “usual” bi-linear form.



### Modified Steklov Formulation - Vertex

☞ The linear forms in the weak formulation are:

$$\mathcal{M}_R(\mathbf{u}, \mathbf{v}) \stackrel{\text{def}}{=} R \iint_{\phi, \theta} [\mathbf{v}^T [n][E][A_\alpha] \mathbf{u}]_R \sin \theta d\theta d\phi$$

and

$$\mathcal{N}_R(\mathbf{u}, \mathbf{v}) \stackrel{\text{def}}{=} R \iint_{\phi, \theta} [\mathbf{v}^T [n][E][D^{(\theta, \phi)}] \mathbf{u}]_R \sin \theta d\theta d\phi$$

☞ Numerical implementation is “on the way”.



### What's there so far for 3-D

- ✎ Edge eigen-pairs (isotropic and anisotropic) can be computed accurately, reliably and efficiently.
- ✎ The modified Steklov weak form for vertex singularities is available - numerical implementation is pending.
- ✎ A new quasidual function method for the computation of edge flux/stress intensity functions is implemented - PPT presentation...