Potential in a polygonal body coated with a thin dielectric layer

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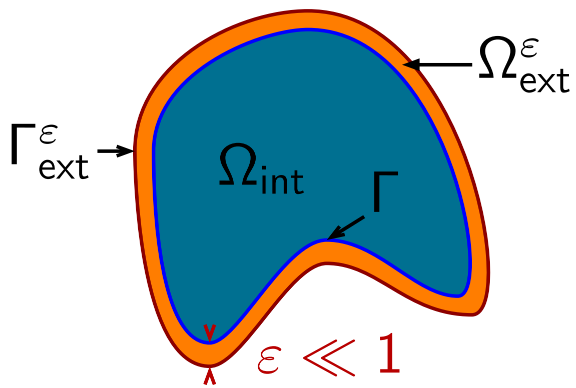
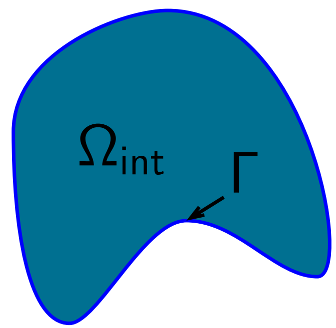
Overview

- Approximate boundary conditions : the smooth case.
- Case of a corner domain : what to do with singularities.
- Numerical experiments.

$$\left\{\begin{array}{rcll} \alpha\Delta u^{\varepsilon}_{int} &=& -f_{int} & \text{in } \Omega_{int}, \\ \Delta u^{\varepsilon}_{ext} &=& 0 & \text{in } \Omega^{\varepsilon}_{ext}, \\ u^{\varepsilon}_{int} &=& u^{\varepsilon}_{ext} & \text{on } \Gamma, \\ \alpha\partial_n u^{\varepsilon}_{int} &=& \partial_n u^{\varepsilon}_{ext} & \text{on } \Gamma, \\ u^{\varepsilon}_{ext} &=& 0 & \text{on } \Gamma^{\varepsilon}_{ext}. \end{array}\right.$$



$$\left\{\begin{array}{ll} \alpha\Delta v^{\varepsilon} = -f_{int} & \text{in } \Omega_{int}, \\ BC(\varepsilon) & \text{on } \Gamma. \end{array}\right.$$



Choice of $BC(\varepsilon)$
such that $u^{\varepsilon}_{int} - v^{\varepsilon} = \mathcal{O}(\varepsilon^p)$?

p	1	2	3
$BC(\varepsilon)$	$v^{\varepsilon} = 0$	$v^{\varepsilon} + \varepsilon\alpha\partial_n v^{\varepsilon} = 0$	$\left(1 + \varepsilon\frac{c(t)}{2}\right) v^{\varepsilon} + \varepsilon\alpha\partial_n v^{\varepsilon} = 0$

$c(t)$ is the curvature on Γ on the point of curvilinear abscissa t.

We perform a dilatation of Ω^ε_{ext} in the normal direction :

$$\begin{array}{ccc} \Omega^\varepsilon_{ext} & \longrightarrow & \Omega^1_{ext} \\ (s,t) & \longmapsto & (\bar{s} = \frac{s}{\varepsilon}, t) \end{array} \qquad (s, t : \text{ Frénet variables}).$$

We define : $\bar{u}^\varepsilon_{ext}(\bar{s}, t) = u^\varepsilon_{ext}(s,t)$.

We consider a fixed domain ; ε appears now in the equations.

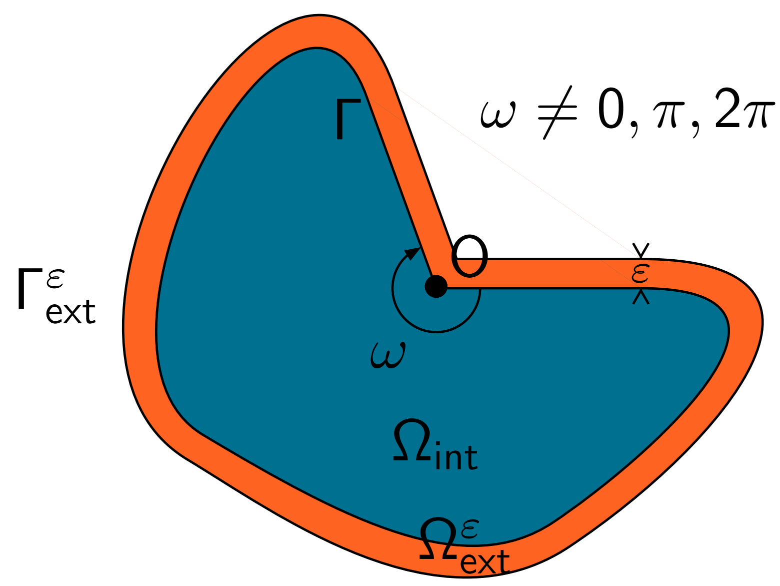
$$\left\{ \begin{array}{rcll} \alpha \Delta u^\varepsilon_{int} & = & -f_{int} & \text{in } \Omega_{int}, \\ \dfrac{1}{\varepsilon^2}\left(\partial^2_{\bar{s}} - \displaystyle\sum_{\ell \geq 1} \varepsilon^\ell A^\ell\right)\bar{u}^\varepsilon_{ext} & = & 0 & \text{in } \Omega^1_{ext}, \\ u^\varepsilon_{int} & = & \bar{u}^\varepsilon_{ext} & \text{on } \Gamma, \\ \alpha \partial_n u^\varepsilon_{int} & = & \frac{1}{\varepsilon}\partial_{\bar{s}} \bar{u}^\varepsilon_{ext} & \text{on } \Gamma, \\ \bar{u}^\varepsilon_{ext} & = & 0 & \text{on } \Gamma^1_{ext}. \end{array} \right.$$

$$\begin{cases} \alpha\Delta u^{\varepsilon}_{int} = -f_{int} & \text{in } \Omega_{int}, \\ \dfrac{1}{\varepsilon^2}\Big(\partial^2_{\bar{s}} - \sum_{\ell\geq 1}\varepsilon^{\ell}A^{\ell}\Big)\bar{u}^{\varepsilon}_{ext} = 0 & \text{in } \Omega^1_{ext}, \\ u^{\varepsilon}_{int} = \bar{u}^{\varepsilon}_{ext} & \text{on } \Gamma, \\ \alpha\partial_n u^{\varepsilon}_{int} = \frac{1}{\varepsilon}\partial_{\bar{s}}\bar{u}^{\varepsilon}_{ext} & \text{on } \Gamma, \\ \bar{u}^{\varepsilon}_{ext} = 0 & \text{on } \Gamma^1_{ext}. \end{cases}$$

The ansätze $u^{\varepsilon}_{int} = \sum_{n\geq 0}\varepsilon^n u^n_{int}$ and $\bar{u}^{\varepsilon}_{ext} = \sum_{n\geq 0}\varepsilon^n \bar{u}^n_{ext}$ lead to the problems

$$\begin{cases} \partial^2_{\bar{s}}\bar{u}^n_{ext} = \sum_{\ell+p=n} A^{\ell}\bar{u}^p_{ext} & \text{in } \Omega^1_{ext}, \\ \partial_{\bar{s}}\bar{u}^n_{ext} = \alpha\partial_n u^{n-1}_{int} & \text{on } \Gamma, \\ \bar{u}^n_{ext} = 0 & \text{on } \Gamma^1_{ext}. \end{cases} \qquad \begin{cases} \alpha\Delta u^n_{int} = -f_{int}\delta^n_0 & \text{in } \Omega_{int}, \\ u^n_{int} = \bar{u}^n_{ext} & \text{on } \Gamma. \end{cases}$$

We obtain the asymptotic expansion of u^{ε} ; by truncature of the series we deduce the approximate boundary conditions.



We perform a dilatation of $\Omega^{\varepsilon}_{ext}$ in the "normal" direction.

Ansätze : $u^{\varepsilon}_{int} = \sum_{n \geq 0} \varepsilon^n u^n_{int}$

$$\bar{u}^{\varepsilon}_{ext} = \sum_{n \geq 0} \varepsilon^n \bar{u}^n_{ext}$$

$$\begin{cases} \partial^2_s \bar{u}^n_{ext} = \sum_{\ell+p=n} A^{\ell} \bar{u}^p_{ext} & \text{in } \Omega^1_{ext}, \\ \partial_{\bar{s}} \bar{u}^n_{ext} = \alpha \partial_n u^{n-1}_{int} + a(t) \partial_t \bar{u}^{n-1}_{ext} & \text{on } \Gamma, \\ \bar{u}^n_{ext} = 0 & \text{on } \Gamma^1_{ext}. \end{cases}$$

$$\begin{cases} \alpha \Delta u^n_{int} = -f_{int} \delta^n_0 & \text{in } \Omega_{int}, \\ u^n_{int} = \bar{u}^n_{ext} & \text{on } \Gamma. \end{cases}$$

- At each step, we loose some regularity.
- Singularities appear, due to the corner.

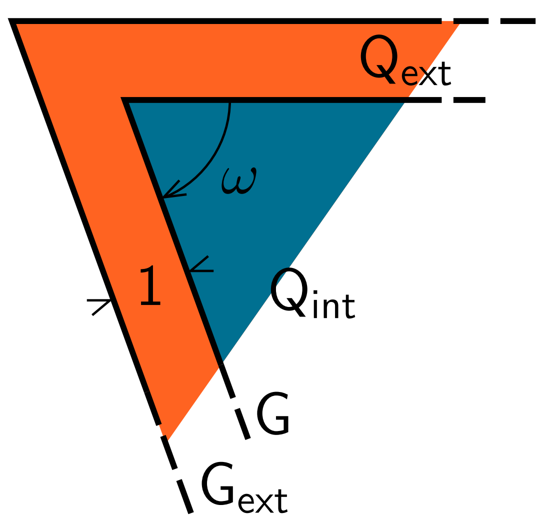
For n = 0 $\quad \bar{u}^0_{ext} = 0 \quad$ and $\quad \begin{cases} \alpha \Delta u^0_{int} = -f_{int} & \text{in } \Omega_{int}, \\ u^0_{int} = 0 & \text{on } \Gamma. \end{cases}$

The regularity of u^0_{int} is limited by the first singular function : $\chi \mathfrak{s}^1(r,\theta)$;

- χ : cut-off function equal to 1 near the corner ;
- $\mathfrak{s}^1(r,\theta) = r^{\frac{\pi}{\omega}} \sin\left(\frac{\pi}{\omega}\theta\right)$, $\quad (r,\theta)$: polar coordinates around the corner.

For n = 1 $\quad \bar{u}^1_{ext} = \alpha \partial_n u^0_{int}(\bar{s} - 1) \notin H^1(\Omega^1_{ext})$ if $\omega \geq \frac{2\pi}{3}$.

Due to the loss of regularity, the singularities cannot be handled with the classical method (exterior-interior successive resolutions).



- f_{int} flat $\Longrightarrow u^0_{int} = \mathbf{u}^0_{flat,int} + \chi \sum_q c^0_q \mathfrak{s}^q(r,\theta) \quad (*)$,

  where $\mathfrak{s}^q(r,\theta) = r^{\frac{q\pi}{\omega}} \sin\left(\frac{q\pi}{\omega}\theta\right)$, $q \geq 1$.

- $\mathfrak{s}^q$ solves the homogeneous Dirichlet problem in Q_{int}.

- Let $\mathfrak{K}^q$ be solution of the homogeneous transmission problem in Q_{int} and Q_{ext} such that $\mathfrak{K}^q \underset{r\to+\infty}{\sim} \mathfrak{s}^q$.

In $(*)$, we replace the singularities

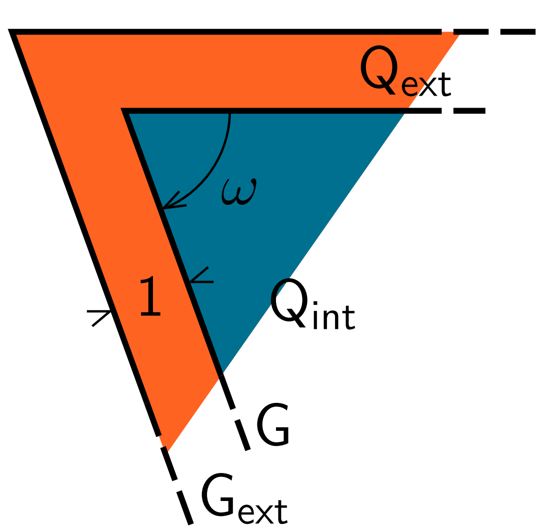
$$u^0_{int} = \mathbf{u}^0_{flat,int} + \chi \sum_q c^0_q \varepsilon^{\frac{q\pi}{\omega}} \mathfrak{s}^q(\tfrac{r}{\varepsilon},\theta) \text{ becomes } \mathbf{u}^0_{flat,int} + \chi \sum_q c^0_q \varepsilon^{\frac{q\pi}{\omega}} \mathfrak{K}^q(\tfrac{r}{\varepsilon},\theta)$$

The part $\mathbf{u}^0_{flat,int}$ is handled with the "exterior-interior" method.

The final asymptotic expansion is

$$u^\varepsilon \approx \sum_\mu \varepsilon^\mu \mathbf{u}^\mu_{flat} + \chi \sum_q \sum_\mu \varepsilon^{\mu+\frac{q\pi}{\omega}} c^\mu_q \mathfrak{K}^q(\tfrac{r}{\varepsilon},\theta) \quad , \mu = p + \frac{k\pi}{\omega} \quad \left| \begin{array}{c} p \in \mathbf{N} \\ k = 0 \text{ or } k \geq 2 \end{array} \right.$$

First exponents : $\mu = 0, 1, \frac{2\pi}{\omega}, \frac{3\pi}{\omega}, 2, \ldots$



$$\begin{cases} \alpha\Delta\mathfrak{K}^{q}_{int} = 0 & \text{in } Q_{int}, \\ \Delta\mathfrak{K}^{q}_{ext} = 0 & \text{in } Q_{ext}, \\ \mathfrak{K}^{q}_{int} = \mathfrak{K}^{q}_{ext} & \text{on } G, \\ \alpha\partial_n\mathfrak{K}^{q}_{int} = \partial_n\mathfrak{K}^{q}_{ext} & \text{on } G, \\ \mathfrak{K}^{q}_{ext} = 0 & \text{on } G_{ext}, \\ \mathfrak{K}^{q} \sim \mathfrak{s}^{q}_{0} & \text{when } r \to \infty. \end{cases}$$

Variational space

$$\mathfrak{V} = \left\{ \mathfrak{v}\ ;\ \frac{\mathfrak{v}}{1+r}, \nabla\mathfrak{v} \in L^2(Q) \text{ and } \mathfrak{v}_{ext} = 0 \text{ on } G_{ext} \right\}.$$

Problem

if $q \geq 0$, $\mathfrak{s}^{q}_{0}$ does not belong to $\mathfrak{V}$!

Over-variational expansion

We construct $\mathfrak{K}^{(0)}, \dots, \mathfrak{K}^{(p)}$ $(p \geq \frac{q\pi}{\omega})$ such that there exists $\mathfrak{v} \in \mathfrak{V}$ and

$$\mathfrak{K}^{q} := \mathfrak{v} + \mathfrak{K}^{(0)} + \cdots + \mathfrak{K}^{(p)} \text{ satisfies the problem.}$$

Under-variational expansion

The Mellin transform $\hat{\mathfrak{v}}(\lambda)$ of $\mathfrak{v}$ is defined for $\operatorname{Re}\lambda > 0$.

- $\hat{\mathfrak{v}}(\lambda)$ has a meromorphic extension κ in $\mathbb{C}$;
- If $\mathcal{M}_\xi^{-1}$ denotes the inverse Mellin transformation along the line $\operatorname{Re}\lambda = \xi$,

$$\mathfrak{v} - \mathcal{M}_\xi^{-1}(r^\lambda \kappa) = \sum \operatorname{Res} r^\lambda \kappa(\lambda).$$

Complete expansion

There exists a solution $\mathfrak{K}^q$ to the problem in the infinite domain such that :

$$\boxed{\mathfrak{K}^{q} = \mathfrak{K}^{(0)} + \cdots + \mathfrak{K}^{(p)} + \mathfrak{K}^{(p+1)} + \cdots + \mathfrak{K}^{(p+N)} + o_\infty\left(r^{\frac{q\pi}{\omega}-p-N}\right).}$$

For the transmission problem

$$u^\varepsilon_{\text{int}} \approx \sum_\mu \varepsilon^\mu u^\mu_{\text{flat,int}} + \chi \sum_q \sum_\mu c^\mu_q \varepsilon^{\mu + \frac{q\pi}{\omega}} \mathfrak{K}^q\left(\frac{r}{\varepsilon}, \theta\right)$$

For the impedance problem $(v^\varepsilon + \varepsilon\alpha\partial_n v^\varepsilon = 0)$

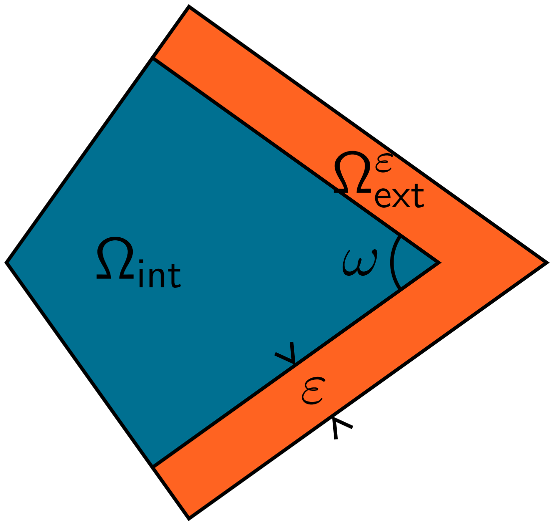
$$v^\varepsilon \approx \sum_\mu \varepsilon^\mu v^\mu_{\text{flat}} + \chi \sum_q \sum_\mu d^\mu_q \varepsilon^{\mu + \frac{q\pi}{\omega}} \mathfrak{Z}^q\left(\frac{r}{\varepsilon}, \theta\right)$$

- For $\mu < \min(3, 1 + \frac{2\pi}{\omega})$, $\quad u^\mu_{\text{flat,int}} = v^\mu_{\text{flat}}$ and $c^\mu_q = d^\mu_q$.
- The profiles satisfy $\quad \mathfrak{K}^q - \mathfrak{Z}^q = \mathcal{O}\left(r^{\max(\frac{q\pi}{\omega} - 3, -\frac{\pi}{\omega})}\right) \quad$ as $r \to +\infty$.

Error estimates

$$\|u^\varepsilon_{\text{int}} - v^\varepsilon\|_{L^2(\Omega_{\text{int}})} = \mathcal{O}\left(\varepsilon^{\min(\frac{2\pi}{\omega}, 1 + \frac{\pi}{\omega}, 3)}\right)$$

$$\|u^\varepsilon_{\text{int}} - v^\varepsilon\|_{H^1(\Omega_{\text{int}})} = \mathcal{O}\left(\varepsilon^{\min(\frac{\pi}{\omega}, 3)}\right)$$



We compute :

- the solution of the transmission problem,
- the solution of the impedance problem.

$\alpha = 10$ and $\operatorname{supp} f_{\text{int}} \subset\subset \Omega_{\text{int}}$.

15 values of ε and 12 values of ω

Computations done with the f.e. library MÉLINA [Daniel Martin, IRMAR]

Using

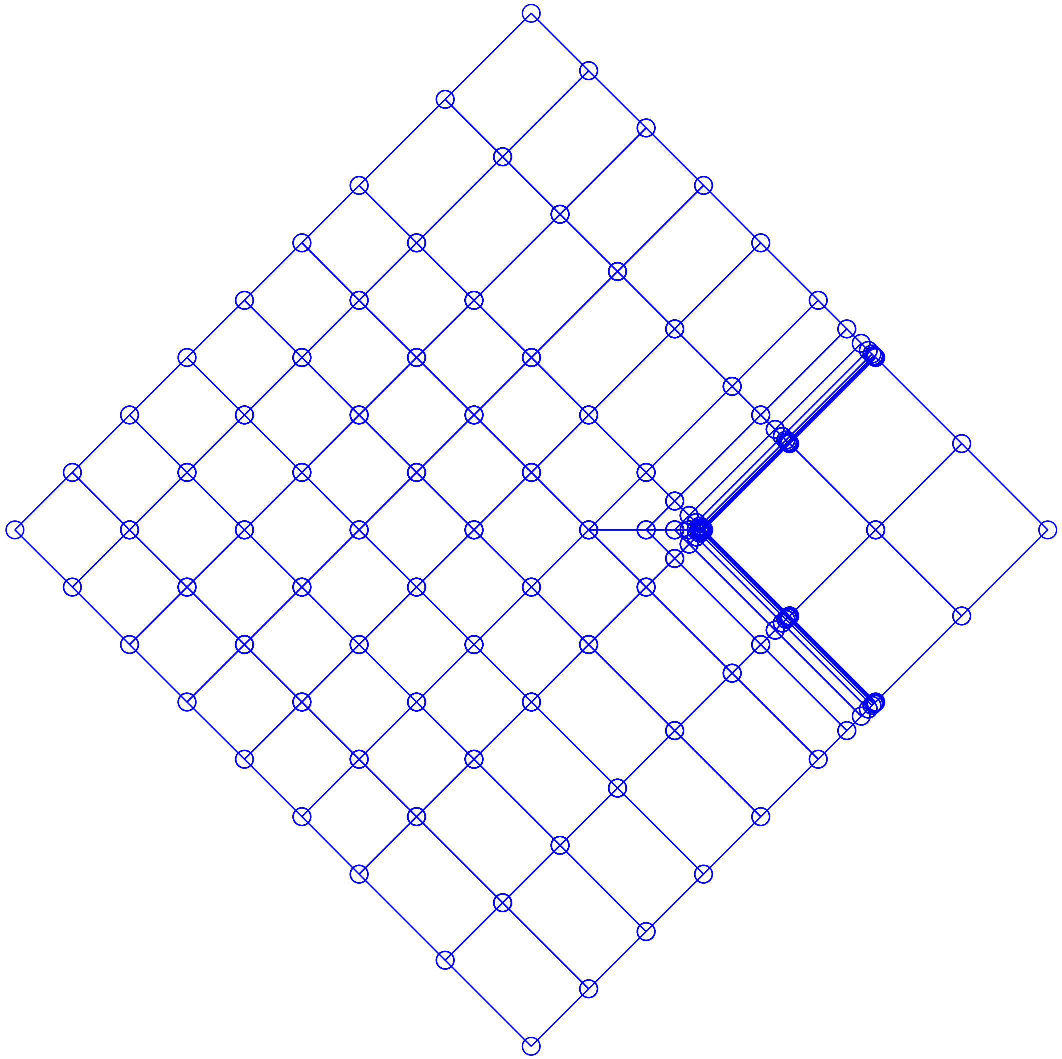
Lagrange $\mathbb{Q}_8$ finite elements,

Geometric refinement near the corner.

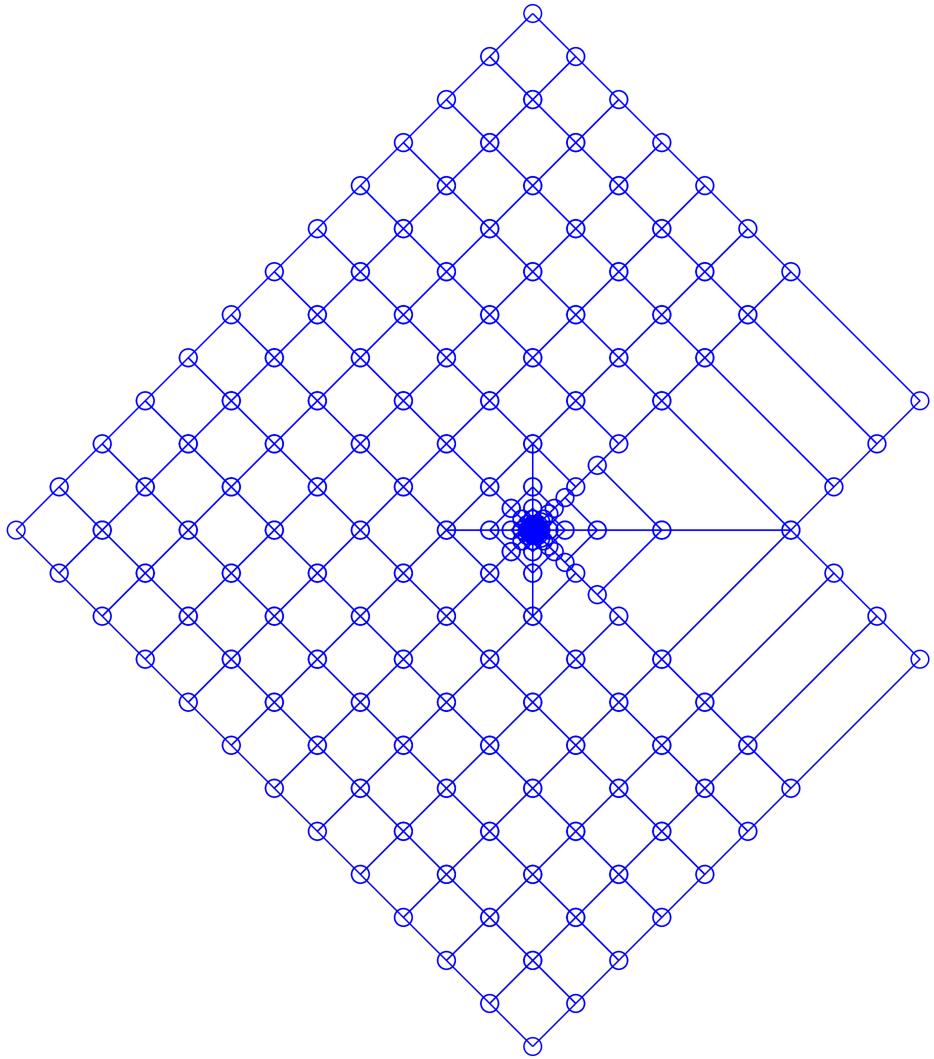
allows

Accuracy of the computation,

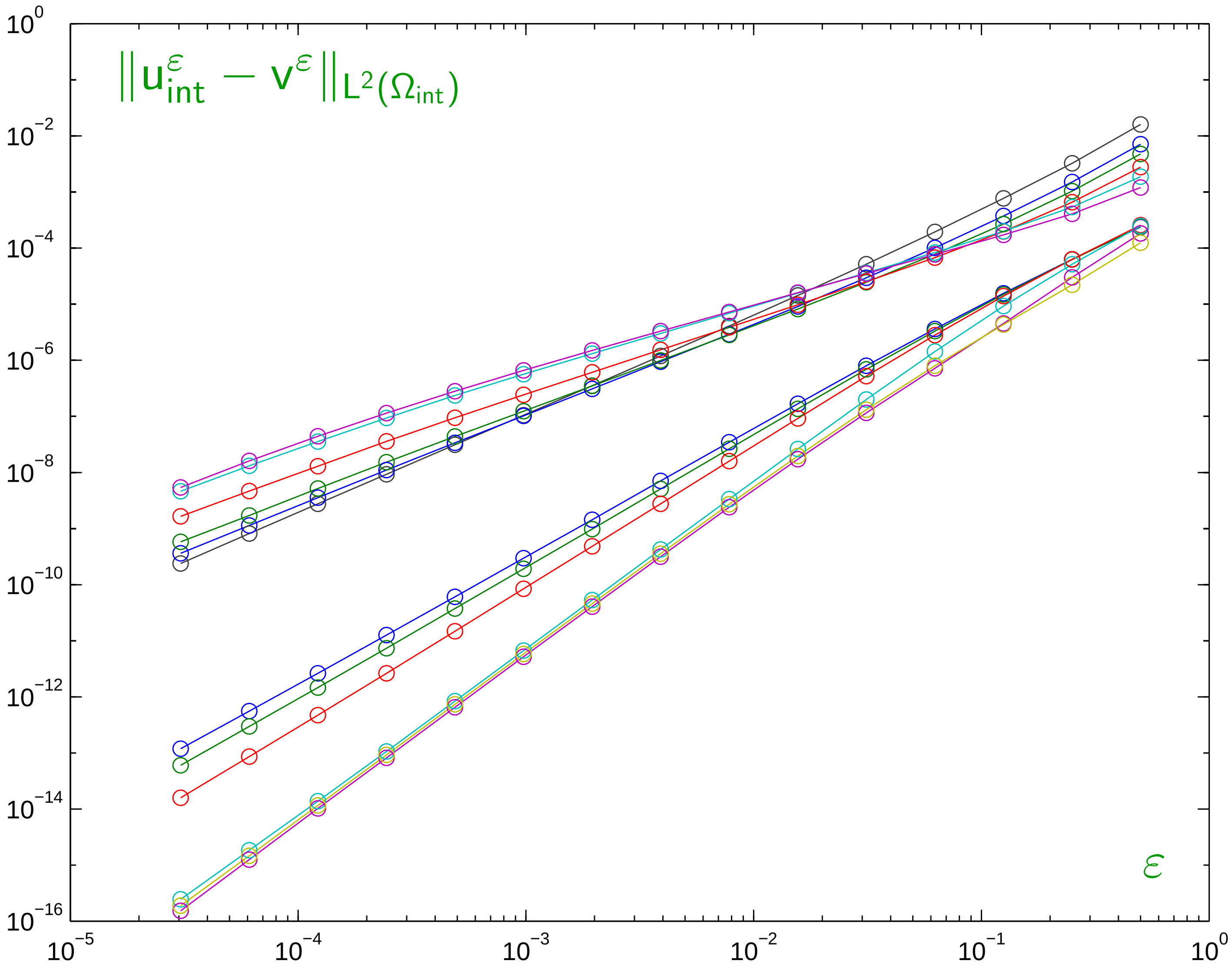
Possible anisotropy in the layer.



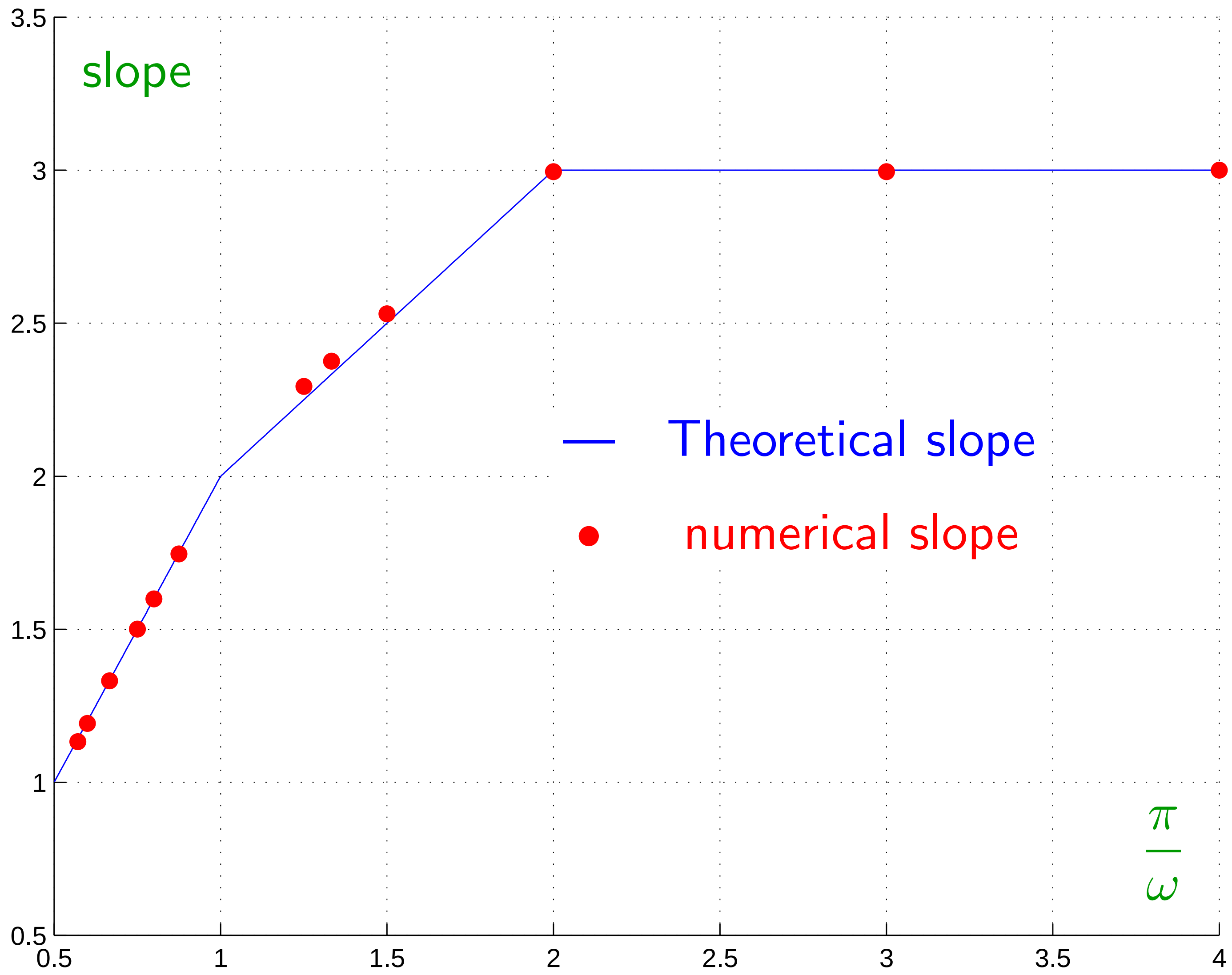
Mesh for $\omega = \frac{\pi}{2}$



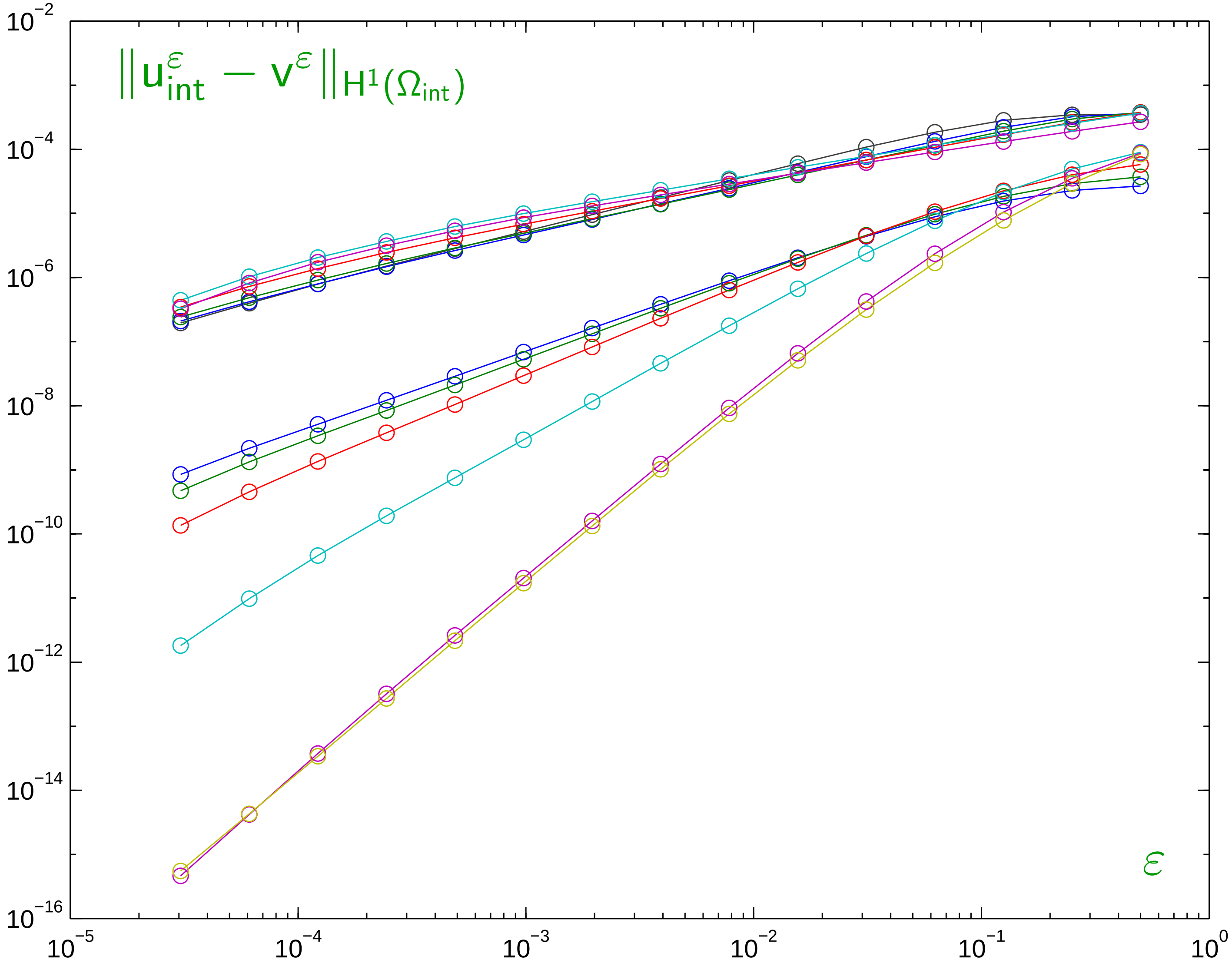
Mesh for $\omega = \frac{3\pi}{2}$



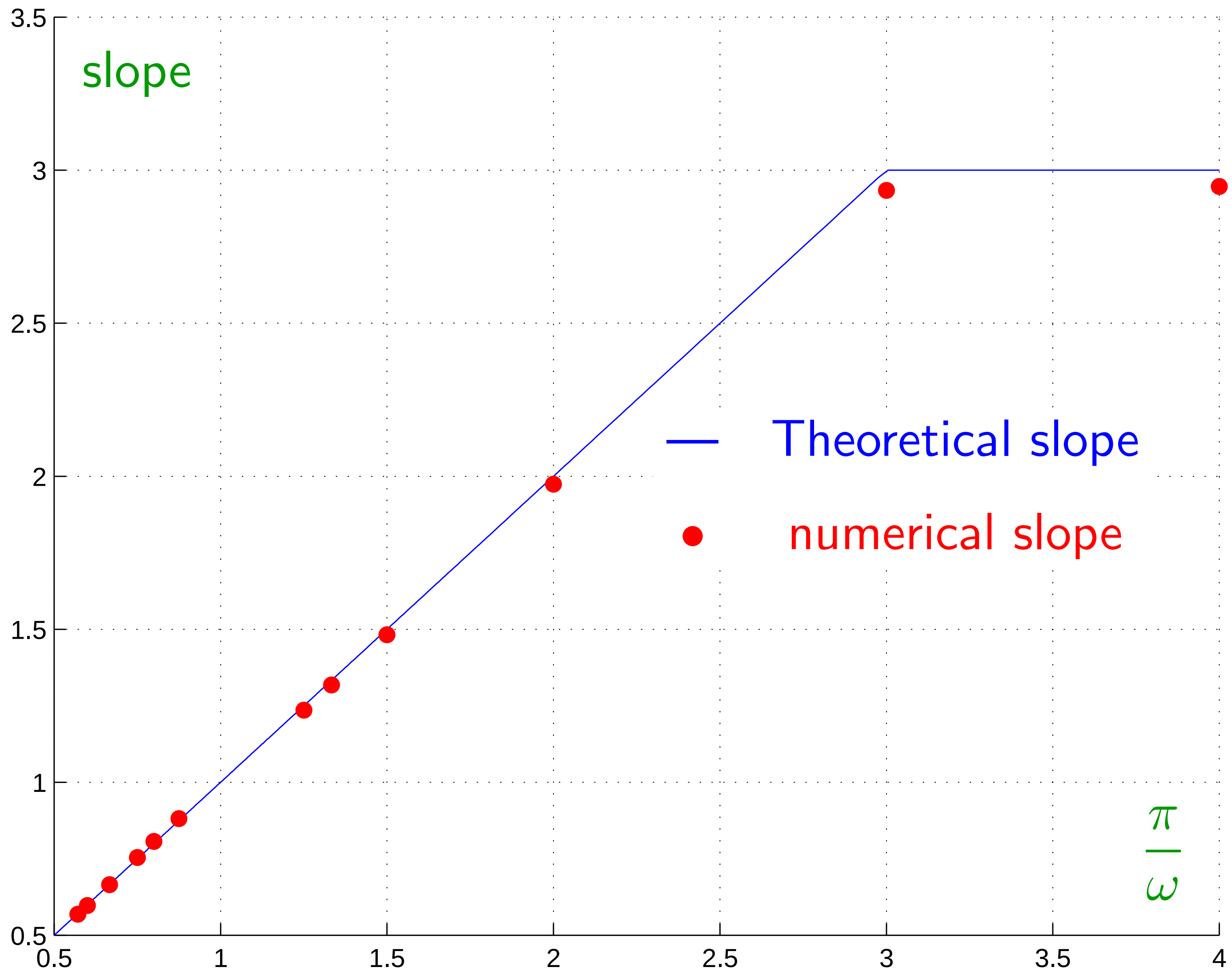
Each "line" corresponds to a value of ω.



Theoretical and numeric slopes (L^2 norm)



Each “line” corresponds to a value of ω.



Theoretical and numeric slopes (H^1 norm)