PDE on multistructures and applications

by

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2-d networks

Def 1 *A 2-d network $\Omega \subset \mathbf{R}^n, n \geq 2$ is a connected set defined by*

$$\Omega = \cup_{i=1}^{I} P_i, I \in \mathbf{N} \setminus \{0\}$$

where

1. P_i is a simply connected polygonal open subset of a plane Π_i,

2. If $i \neq j$, then

$\bar{P}_i \cap \bar{P}_j = \emptyset$, or a common vertex or a whole common edge.

The set $\mathcal{E}$ of edges of Ω is the set of edges of all faces P_i of Ω. The edges are supposed to be open.

If $n = 2$, we say that the network is flat: a special case largely studied in the literature.

On each P_i we fix once and for all a Cartesian system of coordinates $(x_1^{(i)}, x_2^{(i)}) \Rightarrow$ Define Differential operators and function spaces:

Def 2 *Let $\Omega = \cup_{i=1}^{I} P_i$ be a 2-d network. For any $k \geq 1$ we introduce the Sobolev space:*

$$PH^k(\Omega) := \{u = (u_i)_{i=1}^{I} \,:\, u_i \in H^k(P_i)\}.$$

Notation: For a function u defined on Ω we set $u_i = u_{|P_i}$= the restriction of u to P_i.

The Laplace operator

For each i fix a material constant $p_i > 0$. Fix a partition of $\mathcal{E} = \mathcal{D} \cup \mathcal{N}$:

On $\mathcal{D}$: Dirichlet b.c. and on $\mathcal{N}$: Neumann or transmission b.c.

For simplicity we assume that $\mathcal{D} \neq \emptyset$.

The b.v.p.: Given $f \in L^2(\Omega)$, find u solution of

$$\begin{cases} -p_i \Delta u_i = f_i & \text{in } P_i, \forall i, \\ u_i = u_j & \text{on } \partial P_i \cap \partial P_j \text{ (continuity)} \\ u_i = 0 & \text{on } \partial P_i \cap \mathcal{D} \text{ (Dirichlet b.c.)} \\ \sum_{\partial P_i \cap e \neq \emptyset} p_i \frac{\partial u_i}{\partial n_i} = 0 & \text{on } e \in \mathcal{N} \text{ (Neumann or transmission b.c.)} \end{cases}$$

If Ω is flat, then the above problem is a standard interface problem.

Questions: Existence, uniqueness and regularity of the solution.

References: [Kellogg 70], [Carriero 74,76,79],[Ben M'Barek-Mérigot 75], [Lemrabet 77], [Dobrowolski 81], [N. 88], [N.-Dauge 89], [Leguillon-Sanchez Palencia 91], [N.-Sändig 94]

Weak formulation : Introduce

$$\begin{aligned} V &= \{u \in PH^1(\Omega) : u_i = u_j \text{ on } \partial P_i \cap \partial P_j\}, \\ a(u,v) &= \sum_i p_i \int_{P_i} \nabla u_i \cdot \nabla v_i, \forall u, v \in V. \end{aligned}$$

Lax-Milgram lemma $\Rightarrow$: $\exists$ a unique solution $u \in V$ of

$$a(u,v) = \sum_i \int_{P_i} f_i v_i, \forall v \in V. \tag{1}$$

ADN $\Rightarrow$ u is regular far from the corner, i.e., in PH^2.

Corner singularities

Standard ansatz: We look for a singular function S associated with the corner c in the form

$$S^{(\alpha)}(r,\theta) = r^\alpha \varphi(\theta),$$

where (r,θ) are "polar" coordinates centred at c, and satisfying

$$\begin{cases} p_i \Delta S_i = 0 & \text{in } P_i \cap V_c, \forall i, \\ u_i = u_j & \text{on } \partial P_i \cap \partial P_j \cap V_c \\ u_i = 0 & \text{on } \partial P_i \cap \mathcal{D} \cap V_c \\ \sum_{\partial P_i \cap e \neq \emptyset} p_i \frac{\partial u_i}{\partial n_i} = 0 & \text{on } e \in \mathcal{N} \cap V_c. \end{cases}$$

We can see that this is equivalent to the Sturm-Liouville problem on a 1d-network R_c: Find φ defined on R_c and a real number α solution of

$$\begin{cases} \varphi_i'' + \alpha^2 \varphi_i = 0 & \text{in } I_i, \forall i \in \mathcal{I}_c, \\ \varphi_i = \varphi_j & \text{at } \partial I_i \cap \partial I_j \\ \varphi_i = 0 & \text{at } \partial I_i \cap \mathcal{D}_c \\ \sum_{\partial I_i \cap e \neq \emptyset} p_i \frac{\partial u_i}{\partial n_i} = 0 & \text{at } e \in \mathcal{N}_c. \end{cases}$$

Such a α will be called a singular exponant. Let Λ_c= the set of positive singular exponants.

Singular decomposition

Thm 3 *If for all corner c,* $\Lambda_c \cap \{1\} = \emptyset$. *Then*

$$u = u_R + \sum_{\alpha \in \Lambda_c \cap (0,1)} c_\alpha S^{(\alpha)}, \tag{2}$$

where the regular part $u_R \in PH^2(\Omega)$ *and* $c_\alpha \in \mathbf{R}$.

Pf: (Sketch) Using a diadic covering near each corner and a priori estimates, we show that

$$u_i \in H^2_\epsilon(P_i), \forall \epsilon > 0,$$

where $H^2_\epsilon(P_i)$ is the weighted Sobolev space of Kondratiev's type:

$$H^2_\epsilon(P_i) = \{v : r^{\epsilon+|\alpha|-2} D^\alpha v \in L^2(P_i), \forall |\alpha| \leq 2\}.$$

Then apply Mellin transformation as in [Kondratiev 67]. ∎

Coro 4 *[Minimal regularity] If for all corner c,* $\Lambda_c \cap \{1\} = \emptyset$. *Then*

$$u_i \in H^{1+s}(P_i), \forall s > 1 - \alpha, \alpha \in \Lambda_c. \tag{3}$$

Extensions

1. Plate equations [N. 93,94], [Mercier 98]: find u solution of

$$p_i \Delta^2 u_i = f_i \text{ in } P_i, \forall i,$$

+ some continuity and transmission conditions.

2. Lamé system [N.-Sändig 94,99], [Mercier 98]: find a (2d-vector) u solution of

$$L_i u_i = f_i \text{ in } P_i, \forall i,$$

+ some continuity and transmission conditions, where L_i is the Lamé system corresponding to the constitutive material of P_i.

3. Coupled systems [Maghnouji-N. 92], [Mercier 98]: Laplace, plate and/or Lamé equations coupled by some continuity and transmission conditions.

4. Multidimensional problem: An example [Lemrabet 85],[N. 92]

Take

$$\Omega_1 =]-1,1[^2 \backslash \sigma \text{ in } \mathbf{R}^2,\ \sigma = \{(x,0) : 0 < x < 1\},$$
$$\Omega_2 =]0,2[\text{ in } \mathbf{R}.$$

The pb: Find u_1 defined in Ω_1 and u_2 in Ω_2 solution of

$$\begin{cases} -\Delta u_1 = f_1 & \text{in } \Omega_1, \\ -\Delta u_2 + [\frac{\partial u_1}{\partial n}] = f_2 & \text{in } \sigma, \\ u_1 = u_2 & \text{on } \sigma \text{ (continuity)} \\ u_1 = 0 & \text{on } \partial\Omega_1 \setminus \sigma \text{(Dirichlet b.c.)} \\ u_2(0) = u_2(2) = 0 & \text{(Dirichlet b.c.)} \end{cases}$$

$u_1 \sim r^{1/2}$ near $(0,0)$ and $u_2 \sim r^{3/2}$ near 0: Both are singular!

Lower bounds for the eigenvalues of the Laplace equation

For an arbitrary 2d-network [Nicaise 87]:

For a corner c with a Dirichlet edge e containing c and $p_i = 1$, then

$$\alpha_1 \geq \frac{\pi}{2\omega},$$

where $\omega = \sum_{c \in \bar{P}_i} \omega_i$, ω_i being the interior opening of P_i at c.

For corner c with all edge e containing c in $\mathcal{N}$ and $p_i = 1$, then

$$\alpha_1 \geq \frac{\pi}{\omega}.$$

For interface problems [Kuhn 92],[CDN 99],[Petzoldt 00],[Mercier 01]:

2 domains, exterior case (D-D or N-N): $\alpha_1 \geq \frac{\pi}{2\omega}$

2 domains, interior case: $\alpha_1 \geq \frac{1}{2}$

3 domains, interior case: $\alpha_1 \geq \frac{1}{4}$

3 domains, exterior case (D-D or N-N) or 4 domains, interior case: α_1 as small as we want.

n-domains, exterior case (D-D) and the sequence $p_1, \cdots, p_n$ admits only one maximum (quasi-monotone sequence): $\alpha_1 \geq \frac{1}{4}$

n-domains, interior case and a quasi-monotone assumption: $\alpha_1 \geq \frac{1}{4}$

n-domains, exterior case (D-D) and the sequence $p_1, \cdots, p_n$ is decreasing: $\alpha_1 \geq \min(\frac{\pi}{\omega}, \frac{\pi}{2(\omega - \omega_n)})$

All the above estimates are sharp!

More lower bounds in [Mercier 01]

Homotopy argument

Goes back to [Kozlov-Maz'ya 88]

adapted to interface problem in [N. 88], [N.-Sändig 94],[Knees 02].

For Laplace: $\alpha_1 \geq \frac{1}{2}$ under the following (geometrical) assumptions:

2 domains, exterior case (D-D or N-N) and $\omega_1, \omega_2 < \frac{\pi}{2}$

2 domains, exterior case (D-N), $\omega_1 < \frac{\pi}{2}$, $\frac{\pi}{2} < \omega_2 < \pi$ and $\omega = \omega_1 + \omega_2 < \pi$

n domains, exterior case (D-D), the sequence $\frac{1}{p_1}, \cdots, \frac{1}{p_n}$ is quasi-monotone and some geometrical assumptions

For Lamé: $\Re\alpha_1 \geq \frac{1}{2}$ under the following (geometrical) assumptions:

2 domains, exterior case (D-D or N-N) $\omega_1, \omega_2 < \frac{\pi}{2}$ and $(\lambda_2 - \lambda_1)(\mu_2 - \mu_1) \geq 0$

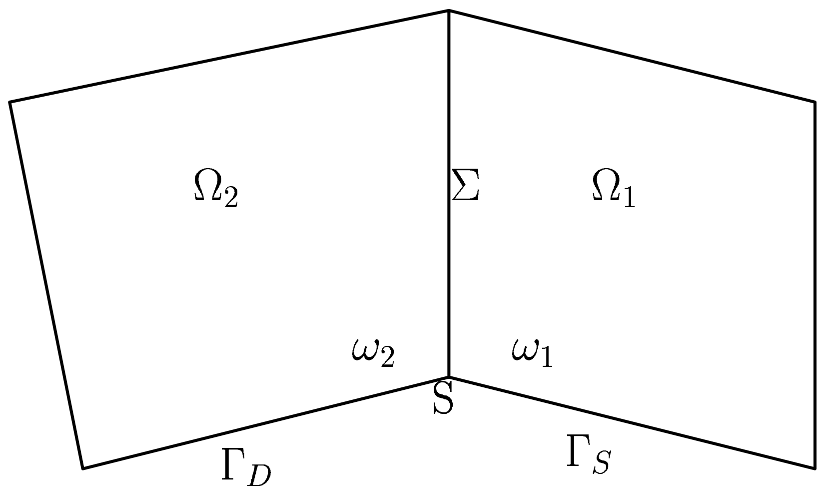
2 domains, exterior case (D-N), $\omega_1 < \frac{\pi}{2}$, $\frac{\pi}{2} < \omega_2 < \pi$, $\omega = \omega_1 + \omega_2 < \pi$ and $(\lambda_2 - \lambda_1)(\mu_2 - \mu_1) \geq 0$

n domains, exterior case (D-D), the sequences $\frac{1}{\lambda_1}, \cdots, \frac{1}{\lambda_n}$ and $\frac{1}{\mu_1}, \cdots, \frac{1}{\mu_n}$ are quasi-monotone and some geometrical assumptions

More results in [Knees 02]

No lower bound are available for a pure interface problem for Lamé.

Signorini transmission problems



Fix β_S and β_Σ two maximal monotone graphs of $\mathbf{R}^2$; $0 \in \beta_S(0)$ and $0 \in \beta_\Sigma(0)$.

For $f \in L^2(\Omega)$, find u solution of

$$(4) \qquad \begin{cases} p_i(-\Delta u_i + u_i) = f_i \text{ in } \Omega_i, i = 1, 2, \\ u_i = u_2 \text{ on } \Sigma, \\ -(p_1 \frac{\partial u_1}{\partial n_1} + p_2 \frac{\partial u_2}{\partial n_2}) \in \beta_\Sigma(u_1) \text{ on } \Sigma, \\ -p_1 \frac{\partial u_1}{\partial n_1} \in \beta_S(u_1) \text{ on } \Gamma_S, \\ u = 0 \text{ on } \Gamma_D. \end{cases}$$

Weak formulation

Variational space: $H^1_D(\Omega) = \{v \in H^1(\Omega) : v = 0 \,\text{on}\, \Gamma_D\}$.

$\beta_S = \partial j_S$ and $\beta_\Sigma = \partial j_\Sigma$: j_S and j_Σ l.s.c functions $\mathbf{R} \to]-\infty, +\infty[$.

We associate two mappings l.s.c. Φ_S and Φ_Σ defined respectively on $L^2(\Gamma_S)$ and $L^2(\Gamma_\Sigma)$ by

$$\Phi_S(u) = \begin{cases} \int_{\Gamma_S} j_S(u)d\sigma, & if\; j_S(u) \in L^1(\Gamma_S), \\ +\infty & else, \end{cases}$$

and

$$\Phi_\Sigma(u) = \begin{cases} \int_{\Gamma_\Sigma} j_\Sigma(u)d\sigma, & if\; j_\Sigma(u) \in L^1(\Gamma_\Sigma), \\ +\infty & else. \end{cases}$$

Find a solution $u \in V$ of

$$(5) \qquad \begin{cases} a(u, v-u) & + & \Phi_S(u_{|\Gamma_S}) - \Phi_S(v_{|\Gamma_S}) \\ & + & \Phi_\Sigma(u_{|\Gamma_\Sigma}) - \Phi_\Sigma(v_{|\Gamma_\Sigma}) \\ & \geq & \int_\Omega f(v-u)dx, \; \forall v \in H^1_D(\Omega). \end{cases}$$

Approximated problems

$$(6) \qquad \begin{cases} p_i(-\Delta u_{\lambda,i} + u_{\lambda,i}) = f_i \text{ in } \Omega_i, i = 1, 2, \\ u_{\lambda,1} = u_{\lambda,2} \text{ on } \Sigma, \\ -(p_1 \frac{\partial u_{\lambda,1}}{\partial n_1} + p_2 \frac{\partial u_{\lambda,2}}{\partial n_2}) = \beta_{\Sigma,\lambda}(u_{\lambda,1}) \text{ in } \Sigma \\ -p_1 \frac{\partial u_{\lambda,1}}{\partial n_1} = \beta_{S,\lambda}(u_{\lambda,1}) \text{ in } \Gamma_S \\ u_\lambda = 0 \text{ in } \Gamma_D \end{cases}$$

where $\lambda > 0$, $\lambda \to 0$ and $\beta_{\Sigma,\lambda}$ (resp. $\beta_{S,\lambda}$) is the Yosida approximation of β_Σ (resp. β_S) given by $\beta_{\Sigma,\lambda} = \lambda^{-1}(Id - (Id + \lambda\beta_\Sigma)^{-1})$ and is a non-decreasing function, uniformly Lipschitz continuous with a Lipschitz constant equal to λ^{-1}.

The weak formulation of (6) is:

$$(7) \qquad \begin{aligned} a(u_\lambda, v) \; &+ \textstyle\int_\Sigma \beta_{\Sigma,\lambda}(u_{\lambda,1}) v_1 ds \\ &+ \textstyle\int_S \beta_{S,\lambda}(u_{\lambda,1}) v_1 ds = \int_\Omega f v dx, \; \forall v \in H^1_D(\Omega). \end{aligned}$$

With standard arguments ([Brézis 72]), we show that (7) has a unique solution $u_\lambda \in H^1_D(\Omega)$, which fulfils

$$(8) \qquad ||u_\lambda||_{1,\Omega} \lesssim ||f||_{0,\Omega}.$$

Regularity of the approximated solution

Adaptation of arguments of [Grisvard 76] in the smooth case to the nonsmooth case.

Localization

$$(9) \qquad \begin{cases} -p_i\Delta u_i = F_i \text{ in } \Omega_i, i = 1,2, \\ u_i = u_2 \text{ on } \Sigma, \\ -(p_1\frac{\partial u_1}{\partial n_1} + p_2\frac{\partial u_2}{\partial n_2}) = \eta\beta_{\Sigma,\lambda}(u_{\lambda,1}) \text{ on } \Sigma \\ -p_1\frac{\partial u_1}{\partial n_1} = \eta\beta_{S,\lambda}(u_{\lambda,1}) \text{ on } \Gamma_S \\ u = 0 \text{ on } \Gamma_D \end{cases}$$

η : radial cut-off function $\eta \equiv \eta(r) \in \mathcal{D}(\mathbf{R}^+)$; $u = \eta u_\lambda$.

$$||F||_{0,C} \lesssim ||f||_{0,\Omega}.$$

Thm 5 *For any* $1 > \delta > 1 - \alpha_1$ *and any* $i, j, k = 1, 2$, *one has*

$$(10) \qquad r^\delta \partial^2_{jk} u_i \in L^2(\Omega_i).$$

Furthermore one has

$$(11) \qquad \sum_{i=1,2} \sum_{j,k=1,2} \|r^\delta \partial^2_{jk} u_i\|_{0,C_i} \lesssim \|f\|_{0,\Omega}.$$

Pf: Since $F \in L^2(\Omega)$, $-\eta\beta_{S,\lambda}(u_{\lambda,1}) \in H^{1/2}(\Gamma_S)$, $-\eta\beta_{\Sigma,\lambda}(u_{\lambda,1}) \in H^{1/2}(\Sigma)$,

$$u = u_R + \sum_{\alpha \in \Lambda \cap (0,1)} c_\alpha S^{(\alpha)},$$

where $u_R \in PH^2(\Omega)$ is the regular part of u, $c_\alpha \in \mathbf{R}$ and $S^{(\alpha)}$ are the singularities of the transmission problem (9).

The estimate (11) is obtained by using some tricky integration by parts and using the monotonicity of $\beta_{\Sigma,\lambda}$ and $\beta_{S,\lambda}$. ∎

Regularity of the solution

Thm 6 *[Chikouche-Mercier-N. 02] Let u be the solution of (4). Then for any $i, j, k = 1, 2$, one has*

$$\rho_i \partial^2_{jk} u_i \in L^2(\Omega_i), \tag{12}$$

where

$$\rho_1 = \prod_1^{N_1} r_j^{\delta_j}, \ \rho_2 = r_1^{\delta_1} \prod_1^{N_1+N_2-2} r_j^{\delta_j},$$

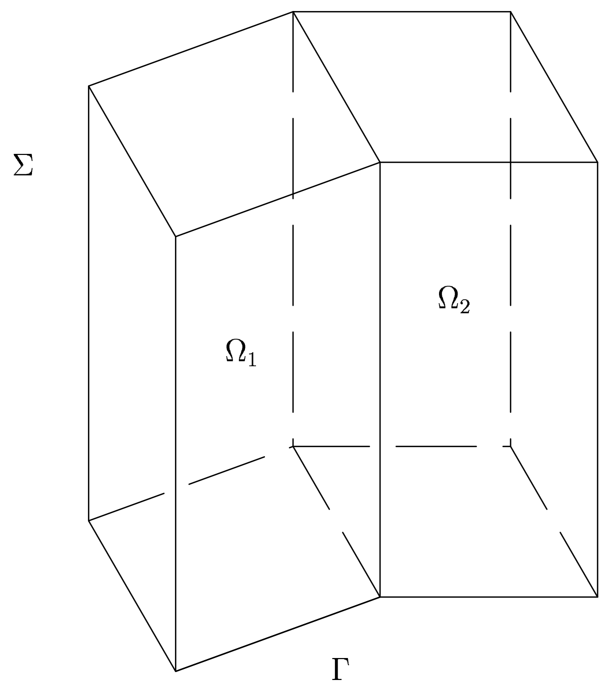
r_j is the distance to P_j and $1 > \delta_j > 1 - \alpha_j$.

Pf: Since $\delta_j \in]0, 1[$, the embedding of $H^{2,\vec{\delta}}(\Omega_i)$ into $H^1(\Omega_i)$ is compact [Grisvard, 85]. Hence the result follows from the uniform estimates on u_λ and a standard passage to the limit. ∎

3d problems: edge singularities

Fix a three dimensional prismatic domain $\Omega = G \times I$, where $I = (0, 1)$ and G is decomposed into two non-overlapping polygonal domains G_1 and G_2 with an interface σ.

This partition induces a similar partition of Ω into two subdomains $\Omega_i = G_i \times I$, the interface being $\Sigma = \sigma \times I$.



The problem

Fix two positive constants $p_i, i = 1, 2$. Consider the interface problem:

$$(13) \qquad \begin{cases} -p_i \Delta u_i = f_i \text{ in } \Omega_i, i = 1, 2, \\ u_1 = u_2 \text{ on } \Sigma, \\ p_1 \frac{\partial u_1}{\partial n_1} + p_2 \frac{\partial u_2}{\partial n_2} = 0 \text{ on } \Sigma, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Weak formulation: find $u \in H_0^1(\Omega)$ solution of

$$\sum_{i=1,2} p_i \int_{\Omega_i} \nabla u_i \cdot \nabla v_i \, dx = \int_\Omega f(x) v(x) \, dx, \forall v \in H_0^1(\Omega).$$

Corner singularity in $G \Rightarrow$ edge singularity in Ω

See [Maz'ya-Plamenevskii 80], [Grisvard 82], [Dauge 88], [Heinrich 93], [Heinrich-N.-Weber 97],[N.-Sändig 99], [CDN 99]

Non tensorial singular decomposition

Thm 7 *[Heinrich-N.-Weber 97] If $f \in L^2(\Omega)$ and $\Lambda \cap \{1\} = \emptyset$, then the solution $u \in H_0^1(\Omega)$ of (13) admits the following decomposition*

$$u = u_R + \sum_{\alpha \in \Lambda \cap (0,1)} (K \star c_\alpha)(r,z) S^{(\alpha)}(r,\theta),$$

where the regular part

$$u_R \in PH^2(G) = \{v \in H^1(G) : u_i \in H^2(G_i), i = 1, 2\},$$

the stress intensity function $c_\alpha \in H^{1-\alpha}(I)$, and $K \star c_\alpha$ means

$$(K \star c_\alpha)(r,z) = \sum_{k=1}^{\infty} c_{\alpha,k} e^{-kr} \varphi_k(z),$$

when $\varphi_k(z) = \frac{\sqrt{2}}{2} \sin(k\pi z)$ and $c_\alpha = \sum_{k=1}^{\infty} c_{\alpha,k}\varphi_k(z)$.

Pf: We use Fourier series expansions:

$$\begin{aligned} u(x,y,z) &= \sum_{k=1}^{\infty} u_k(x,y)\varphi_k(z), \\ f(x,y,z) &= \sum_{k=1}^{\infty} f_k(x,y)\varphi_k(z). \end{aligned}$$

and remark that each u_k belongs to $H_0^1(G)$ and is the unique solution of the Helmoltz/transmission problem

$$\begin{cases} p_i(-\Delta_2 u_{k,i} + k^2 u_{k,i}) = f_{k,i} \text{ in } G_i, i = 1, 2, \\ u_{k,1} = u_{k,2} \text{ on } \sigma, \\ p_1 \frac{\partial u_{k,1}}{\partial \nu_1} + p_2 \frac{\partial u_{k,2}}{\partial \nu_2} = 0 \text{ on } \sigma, \\ u_k = 0 \text{ on } \partial G. \end{cases}$$

Since u_k may be seen as the solution of

$$-p_i \Delta_2 u_{k,i} = f_{k,i} - p_i k^2 u_{k,i} \text{ in } G_i, i = 1, 2,$$

and this right-hand side belongs to $L^2(G)$, by the 2d results

$$u_k = u_{kR} + \sum_{\alpha \in \Lambda \cap (0,1)} c_{\alpha,k} S^{(\alpha)}(r, \theta),$$

where the regular part u_{kR} belongs to $PH^2(G)$ and $c_{\alpha,k} \in \mathbf{R}$ with

$$\|u_{kR}\|_{p2,G} + \sum_{\alpha \in \Lambda \cap (0,1)} |c_{\alpha,k}| \leq C \sum_{i=1,2} \|f_{k,i} - p_i k^2 u_{k,i}\|_{0,G_i},$$

where here and below C is a positive constant independent of k. But we may show that

$$|c_{\alpha,k}| \leq C k^{\alpha - 1} \|f_k\|_{0,G}, \forall \alpha \in \Lambda \cap (0, 1).$$

The factor e^{-kr} in front of $S^{(\alpha)}(r, \theta)$ is introduced to have the uniform estimate

$$\|u_{kR}\|_{p2,G} + k^2 \|u_{kR}\|_{0,G} \leq C \|f_k\|_{0,G}.$$

We conclude by Fourier synthesis. ■

Applications to a Fourier-refined FEM in [Heinrich-N.-Weber 00].

Tensorial singular decomposition

We don't use the factor $e^{-kr} \Rightarrow$

Thm 8 *[Brenner-N.-Sung 02] If $f \in L^2(\Omega)$ and $\Lambda \cap \{1\} = \emptyset$, then the solution $u \in H_0^1(\Omega)$ of (13) admits the following decomposition*

$$u = u_R + \sum_{\alpha \in \Lambda \cap (0,1)} c_\alpha(z) S^{(\alpha)}(r, \theta), \tag{14}$$

where the regular part $u_R \in L^2(I; PH^2(G))$ and $c_\alpha \in H^{1-\alpha}(I)$.

Drawback of the decomposition (14): u_R has not the optimal regularity $PH^2(\Omega)$.

Remedy: assume an additional edge regularity on f.

Thm 9 *[Brenner-N.-Sung 02] Take the assumptions of Theorem 8 and furthermore $f \in H^2(I; L^2(G)) \cap H_0^1(I; L^2(G))$. Then the decomposition (14) still holds but with $u_R \in PH^2(\Omega)$ and $c_\alpha \in H^2(I) \cap H_0^1(I)$.*

Pf: Follows from the additional regularity of f and Parseval's identity. ∎

Applications to multigrid methods in [Brenner-N.-Sung 02].

Applications

1. 3d Signorini's problem [Chikouche-Mercier-N. 02]: Regularity of the solution in term of weighted spaces and optimal regularity in the edge direction.

2. Maxwell interface problem with standard b.c. [CDN 99]: Consider

$$\begin{cases} \mathbf{curl}\, H + i\omega\varepsilon E = J \text{ in } \Omega, \\ \mathbf{curl}\, E - i\omega\mu H = J \text{ in } \Omega, \\ \mathbf{div}\,(\varepsilon E) = \mathbf{div}\,(\mu H) = 0 \text{ in } \Omega, \\ E \times \nu = 0 \text{ and } H \cdot n = 0 \text{ on } \partial\Omega, \end{cases}$$

where ε and μ are piecewise constant.
Main singularities of this problem are

$$\nabla S_\varepsilon^{Dir} \text{ for } E,\ \nabla S_\mu^{Neu} \text{ for } H,$$

where S_ε^{Dir} is the singularity of the Laplace interface problem with coefficients ε and Dirichlet boundary conditions.

3. Maxwell interface problem with impedance b.c. [Lohrengel-N. 02]
The b.c. are replaced by

$$n \times (E \times n) = -i\lambda H \times n \text{ on } \partial\Omega,$$

for some complex number λ such that $\Im\lambda \neq 0$.
Show some density results related to this problem, also related to $\nabla S_\varepsilon^{Dir}$

4. Inverse problems [N.-Zaïr 01]: Determine one emerging crack by one measurement on a part of the boundary.

5. Control problems: Singular behaviour of the solution $\Rightarrow$ either some geometrical conditions or add some internal control near the singular points.

6. Numerical schemes: Singular behaviour of the solution $\Rightarrow$ use some adapted methods to restore the optimal order of convergence.