Density results in the context of the time-harmonic Maxwell equations

Stephanie Lohrengel

Laboratoire J.A. Dieudonné UMR-CNRS 6621

Université de Nice-Sophia Antipolis, France

Email: lohrenge@math.unice.fr

Density results

- . . . are necessary to get equivalence between weak and strong formulations
- . . . are necessary to make H^1-conforming FE-approximations converge to the exact solution

Density results in the context of the classical Maxwell equations

Perfect conducting boundary condition

- Functional space:

$$\mathcal{H}_0(\mathrm{curl}) = \left\{ \boldsymbol{U} \in L^2(\Omega)^3 \;\middle|\; \mathbf{curl}\, \boldsymbol{U} \in L^2(\Omega)^3; (\boldsymbol{U} \times \boldsymbol{n})_{|\Gamma} = 0 \right\}$$

- Density result: $\mathcal{D}(\Omega)^3$ is dense in $\mathcal{H}_0(\mathrm{curl})$.

Impedance boundary condition

- Functional space:

$$W = \left\{ \boldsymbol{U} \in L^2(\Omega)^3 \;\middle|\; \mathbf{curl}\, \boldsymbol{U} \in L^2(\Omega)^3; (\boldsymbol{U} \times \boldsymbol{n})_{|\Gamma} \in L^2(\Gamma)^3 \right\}$$

- Density result: $\mathcal{D}(\overline{\Omega})^3$ is dense in W [Ben Belgacem *et. al.*, '97].

Outline

A regularized formulation of the Maxwell equations

I. Perfect conducting boundary – homogeneous body

II. Perfect conducting boundary – composite material

III. Impedance boundary condition – homogeneous body

IV. Impedance boundary condition – composite material

The regularized time-harmonic Maxwell equations

classical problem

$$\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}) - \omega^2\varepsilon\boldsymbol{E} = \boldsymbol{J} \qquad \text{in } \Omega$$

$$\operatorname{div}(\varepsilon\boldsymbol{E}) = 0 \qquad \text{in } \Omega$$

$+$ boundary conditions on Γ

“regularized” problem

$$\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\boldsymbol{E}) - \varepsilon\,\mathbf{grad}\operatorname{div}(\varepsilon\boldsymbol{E}) - \omega^2\varepsilon\boldsymbol{E} = \boldsymbol{J} \qquad \text{in } \Omega$$

$$\operatorname{div}(\varepsilon\boldsymbol{E}) = 0 \qquad \text{on } \Gamma$$

$+$ same boundary conditions on Γ

I. Perfect conducting boundary – homogeneous body

- Functional space:

$$\mathcal{H}_N(\mathrm{curl}, \mathrm{div}) = \left\{ \boldsymbol{U} \in \mathcal{H}(\mathrm{curl}) \;\middle|\; \mathrm{div}\, \boldsymbol{U} \in L^2(\Omega);\ (\boldsymbol{U} \times \boldsymbol{n})_{|\Gamma} = 0 \right\}$$

- Regular fields:

$$\mathcal{H}^1_N(\Omega) = \left\{ \boldsymbol{U} \in H^1(\Omega)^3 \;\middle|\; (\boldsymbol{U} \times \boldsymbol{n})_{|\Gamma} = 0 \right\}$$

More regular fields ($1 < s \leq \infty$):

$$\mathcal{H}^s_N(\Omega) = \left\{ \boldsymbol{U} \in H^s(\Omega)^3 \;\middle|\; (\boldsymbol{U} \times \boldsymbol{n})_{|\Gamma} = 0 \right\}$$

- Density result (for Ω Lipschitz-polyhedron):

$\mathcal{H}^\infty_N(\Omega)$ is dense in $\mathcal{H}^1_N(\Omega)$

- Ω of class $\mathcal{C}^{1,1}$ or convex Lipschitz-domain:

$$\mathcal{H}_N^1(\Omega) \equiv \mathcal{H}_N(\mathrm{curl}, \mathrm{div})$$

- Decomposition theorem for Ω non-convex Lipschitz-polyhedron [a]:

$$\mathcal{H}_N(\mathrm{curl}, \mathrm{div}) = \mathcal{H}_N^1(\Omega) \oplus \mathcal{H}_{\mathrm{sing}}$$

$\Rightarrow$ Lack of density ($\mathcal{H}_{\mathrm{sing}} \neq \{0\}$).

[a] see [Assous/Ciarlet/Sonnendrücker., '98], [Birman/Solomyak, '87], [Bonnet-BenDhia/Hazard/L., '99], [Costabel/Dauge, 2000]

Existence of a regular vector potential

Let $\boldsymbol{E} \in \mathcal{H}_N(\mathrm{curl}, \mathrm{div})$. There is $\boldsymbol{E}_R \in H^1(\Omega)^3$ such that

$$\mathbf{curl}\, \boldsymbol{E}_R = \mathbf{curl}\, \boldsymbol{E} \quad \text{in } \Omega$$

$$\boldsymbol{E}_R \times \boldsymbol{n} = 0 \quad \text{on } \Gamma$$

$\Rightarrow \quad \boldsymbol{E} = \boldsymbol{E}_R + \mathbf{grad}\, p$ with

$$p \in \mathcal{D}(\Delta^{\mathsf{Dir}}) \stackrel{\mathsf{def}}{=} \{ p \in H^1_0(\Omega) \,\big|\, \Delta p \in L^2(\Omega) \}$$

$\Rightarrow$ All elements of $\mathcal{H}_{\mathrm{sing}}$ derive from the singularities of the scalar Laplacien with Dirichlet b. c.

Consequences: Nodal (H^1-conforming) FE-approximations converge to a field in $\mathcal{H}^1_N(\Omega)$, but not to the physical solution.

$\Rightarrow$ mesh refinement must fail

$\Rightarrow$ Close the gap by adding explicitly singular fields that span $\mathcal{H}_{\text{sing}}$ (in 2D: dim$\mathcal{H}_{\text{sing}} = \#$ reentrant corners) [a]

$\Rightarrow$ Use a weighted regularization term to overcome the lack of density[b]

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[a] see [Assous *et. al.* '98], [Hazard/L. '02]

[b] see [Costabel *el. al.* '01]

II. Perfect conducting boundary – composite material

Propagation domain:

$\Omega \subset \mathbb{R}^d$, $d = 2, 3$ (bounded) polygon or Lipschitz-polyhedron

Electromagnetic coefficients: ε, μ piecewise constant functions on Ω

$\Rightarrow$ Partition $\mathcal{P} = (\Omega_j)_{j=1}^J$ with Ω_j polygons/polyhedra

- Set of exterior (resp. interior) faces: $\mathcal{F}_{ext}$, $\mathcal{F}_{int}$
- Set of exterior (resp. interior) vertices: $\mathcal{S}_{ext}$, $\mathcal{S}_{int}$
- Set of exterior (resp. interior) edges (in 3D): $\mathcal{E}_{ext}$, $\mathcal{E}_{int}$

- Functional space:

$$\mathcal{H}_N(\mathrm{curl}, \mathrm{div}\, \varepsilon) = \left\{ \boldsymbol{U} \in \mathcal{H}(\mathrm{curl}) \,\middle|\, \mathrm{div}(\varepsilon \boldsymbol{U}) \in L^2(\Omega);\ (\boldsymbol{U} \times \boldsymbol{n})_{|\Gamma} = 0 \right\}$$

- Space of piecewise regular fields ($1 \leq s \leq \infty$):

$$P\boldsymbol{H}^s(\Omega; \mathcal{P}) = \left\{ \boldsymbol{U} : \Omega \to \mathbb{R}^3 \,\middle|\, \boldsymbol{U}_{|\Omega_j} \in H^s(\Omega_j)^3; \Omega_j \in \mathcal{P} \right\}$$

- Density result[a]: $(P\boldsymbol{H}^\infty(\Omega; \mathcal{P}) \cap \mathcal{H}_N(\mathrm{curl}, \mathrm{div}\, \varepsilon))$ is dense in $\big(P\boldsymbol{H}^1(\Omega; \mathcal{P}) \cap \mathcal{H}_N(\mathrm{curl}, \mathrm{div}\, \varepsilon)\big)$

- Decomposition theorem[a]:

$$\mathcal{H}_N(\mathrm{curl}, \mathrm{div}\, \varepsilon) = \big(P\boldsymbol{H}^1(\Omega; \mathcal{P}) \cap \mathcal{H}_N(\mathrm{curl}, \mathrm{div}\, \varepsilon)\big) \oplus \mathcal{H}_{\mathrm{sing}}(\varepsilon)$$

$\Rightarrow$ Lack of density

[a] see [Costabel/Dauge/Nicaise, '99]

Existence of a regular vector potential

Let $\boldsymbol{E} \in \mathcal{H}_N(\mathrm{curl}, \mathrm{div}\,\varepsilon)$:

- There is $\boldsymbol{F}_R \in H^1_N(\Omega)$ such that

$$\mathbf{curl}\,\boldsymbol{F}_R = \mathbf{curl}\,\boldsymbol{E} \text{ in } \Omega.$$

Thus $\boldsymbol{E} = \boldsymbol{F}_R + \mathbf{grad}\,p$ with

$$p \in H^1_0(\Omega), \text{ but } \mathrm{div}(\varepsilon\,\mathbf{grad}\,p) \notin L^2(\Omega).$$

- There is $\boldsymbol{E}_R \in P\boldsymbol{H}^1(\Omega;\mathcal{P}) \cap \mathcal{H}_N(\mathrm{curl}, \mathrm{div}\,\varepsilon)$ such that

$$\mathbf{curl}\,\boldsymbol{E}_R = \mathbf{curl}\,\boldsymbol{E} \text{ in } \Omega.$$

Thus $\boldsymbol{E} = \boldsymbol{E}_R + \mathbf{grad}\,p$ where

$$p \in \mathcal{D}(\Delta_\varepsilon^{\mathsf{Dir}}) \stackrel{\mathsf{def}}{=} \left\{p \in H^1_0(\Omega) \,\middle|\, \mathrm{div}(\varepsilon\,\mathbf{grad}\,p) \in L^2(\Omega)\right\}.$$

III. Impedance boundary condition – homogeneous body

$$\Rightarrow \boldsymbol{n} \times (\boldsymbol{E} \times \boldsymbol{n}) = -\lambda^{-1}(\mu^{-1}\,\mathrm{curl}\,\boldsymbol{E} \times \boldsymbol{n})$$

- Functional space:

$$W = \left\{ \boldsymbol{U} \in \mathcal{H}(\mathrm{curl}) \,\middle|\, \mathrm{div}\,\boldsymbol{U} \in L^2(\Omega);\ (\boldsymbol{U} \times \boldsymbol{n})_{|\Gamma} \in L^2(\Gamma) \right\}$$

- Space of regular fields: $H^1(\Omega)^3$
- Density theorem [a]: $H^1(\Omega)^3$ is dense in W for any Lipschitz domain

$\Rightarrow$ No lack of density !

[a] see [Ciarlet Jr./Hazard/L., '98], [Costabel/Dauge, '98]

Existence of a regular vector potential

Let $\boldsymbol{E} \in W$. There is $\boldsymbol{E}_R \in H^1(\Omega)^3$ such that

$$\mathbf{curl}\, \boldsymbol{E}_R = \mathbf{curl}\, \boldsymbol{E} \text{ in } \Omega$$

$$\operatorname{div} \boldsymbol{E}_R = 0 \text{ in } \Omega$$

Then $\boldsymbol{E} = \boldsymbol{E}_R + \mathbf{grad}\, p$ where

$$p \in H \stackrel{\text{def}}{=} \{p \in H^1(\Omega) \mid \Delta p \in L^2(\Omega); \partial_n p \in L^2(\Gamma)\}$$

$$\Leftrightarrow p \in H^1(\Gamma)$$ [a]

[a] see [Jerison/Kenig, '81]

Density result for scalar potentials

Two different techniques

- *Via* the regularity of the scalar potential for arbitrary domains with Lipschitz boundary[a]:

$$H \equiv \left\{ p \in H^{3/2}(\Omega) \,\middle|\, \Delta p \in L^2(\Omega) \right\}$$

and

$$\mathcal{C}^\infty(\overline{\Omega}) \text{ is dense in } H.$$

[a] [Costabel/Dauge, '98]

• *Via* the identification of the orthogonal of $\overline{H^2(\Omega)}$ in H when Ω is a Lipschitz-polyhedron [a]:

Norm for $H/\mathbb{R}$:

$$|u|_H \stackrel{\text{def}}{=} \left(\|\Delta u\|_{0,\Omega}^2 + \|\partial_n u\|_{0,\Gamma}^2\right)^{1/2}$$

Characterization of $f \in \overline{H^2(\Omega)/\mathbb{R}}^{\perp}$:

$$(\Delta f, \Delta u)_\Omega + (\partial_n f, \partial_n u)_\Gamma = 0 \ \forall u \in H^2(\Omega)$$

In particular, with $g \stackrel{\text{def}}{=} \Delta f$:

$$(g, \Delta u)_\Omega = 0 \ \forall u \in H^2(\Omega) \text{ such that } (\partial_n u)_{|\Gamma} = 0$$

$\Rightarrow g$ is a dual singularity of the Laplacian with Neumann b. c.

[a] [Ciarlet Jr./Hazard/L., 98]

Thus

$$\Delta g = 0 \text{ in } \Omega$$

$$\partial_n g = 0 \text{ weakly on } \Gamma_j$$

But

$$\begin{aligned}(\partial_n f, \partial_n u)_\Gamma &= -(g, \Delta u)_\Omega \\ &= -(\Delta g, u)_\Omega + \sum_{\Gamma_j \in \Gamma} \left(< \partial_n g, u >_{\Gamma_j} - < g, \partial_n u >_{\Gamma_j} \right)\end{aligned}$$

and thus

$$g_{|\Gamma_j} = -\partial_n f \in L^2(\Gamma_j)$$

$\Rightarrow$ The elements of the orthogonal of $\overline{H^2(\Omega)}$ are dual singularities with supplementary boundary regularity !

In 2D near a corner with opening angle $\omega > \pi$:

$$g(r,\theta) = Ar^{-\pi/\omega}\cos(\pi\theta/\omega) + g_R.$$

On the boundary ($\theta = 0$):

$$(g - g_R)_{|\{\theta=0\}} = Ar^{-\pi/\omega} \in L^2(0,R)$$

$\Rightarrow A = 0$

$\Rightarrow g$ is the variational solution in $H^1(\Omega)/\mathbb{R}$ of the homogeneous Neumann problem, i. e. $g \equiv 0$.

$\Rightarrow$ Density.

IV. Impedance boundary condition – composite material

- Functional space:

$$\mathsf{W}_\varepsilon = \left\{ \boldsymbol{U} \in \mathcal{H}(\mathrm{curl}) \,\middle|\, \mathrm{div}(\varepsilon \boldsymbol{U}) \in L^2(\Omega);\ (\boldsymbol{U} \times \boldsymbol{n})_{|\Gamma} \in L^2(\Gamma) \right\}$$

- Space of regular fields:

$$P\boldsymbol{H}^1(\Omega; \mathcal{P}) \cap \mathsf{W}_\varepsilon$$

If ε satisfies certain conditions $\Rightarrow$ Density theorem,

otherwise $\Rightarrow$ Decomposition theorem[a]

[a] see [L./Nicaise, '00], [L./Nicaise, '02]

Singularities of the scalar transmission operator (2D)

Standard singularities of $\Delta_\varepsilon^{\mathsf{Dir}}$ at vertex $S \in \mathcal{S}_{ext} \cup \mathcal{S}_{int}$:

$$S_{S,\lambda}(r,\theta) = \eta_S(r) r^\lambda \Phi_\lambda(\theta),\ \ \lambda \in \Lambda_{\varepsilon,S}^{\mathsf{Dir}}$$

Set of singular exponents: $\lambda \in \Lambda_{\varepsilon,S}^{\mathsf{Dir}}$ if and only if there is a non-trivial solution $\Phi_\lambda = \{\Phi_{\lambda,j}\}_j^{J_S}$ of the problem

$$\partial_\theta^2 \Phi_{\lambda,j} + \lambda^2 \Phi_{\lambda,j} = 0; \quad \theta \in]\sigma_{j-1}, \sigma_j[$$

$$\Phi_{\lambda,j}(0) = \Phi_{\lambda,j}(\sigma_{J_S}) = 0 \text{ if } S \in \mathcal{S}_{ext}$$

$$\Phi_{\lambda,j}(0) = \Phi_{\lambda,j}(2\pi) \text{ and } \varepsilon_1 \Phi'_{\lambda,j}(0) = \varepsilon_{J_S} \Phi'_{\lambda,j}(2\pi) \text{ if } S \in \mathcal{S}_{int}$$

$$+ \text{ transmission conditions at } \sigma_j$$

In a similar way, define the set of singular exponents $\Lambda_{\varepsilon,S}^{\mathsf{Neu}}$ of the transmission operator with Neumann b. c., $\Delta_\varepsilon^{\mathsf{Neu}}$.

Existence of a regular vector potential

Technical assumption: $1 \notin \Lambda^{\mathsf{Neu}}_{\varepsilon,S}$

Let $\boldsymbol{E} \in \mathsf{W}_\varepsilon$. There is $\boldsymbol{E}_R \in H^1(\Omega)^3$ such that

$$\mathbf{curl}\, \boldsymbol{E}_R = \mathbf{curl}\, \boldsymbol{E} \quad \text{in } \Omega$$

$$\boldsymbol{E}_R \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma$$

$$\|\boldsymbol{E}_R\|_{P\boldsymbol{H}^1(\Omega;\mathcal{P})} \leq c\|\boldsymbol{E}\|_{\mathsf{W}_\varepsilon}$$

Thus $\boldsymbol{E} = \boldsymbol{E}_R + \mathbf{grad}\, p$ with

$$p \in H_\varepsilon \stackrel{\text{def}}{=} \left\{u \in H^1(\Omega) \,\middle|\, \Delta_\varepsilon u \in L^2(\Omega);\ u_{|\Gamma} \in H^1(\Gamma);\ l(u_{|\Gamma}) = 0\right\}$$

l linear form on $H^1(\Gamma)$

$l : \varphi \mapsto \sum_{S \in \mathcal{S}_{ext}} \varphi(S)$

Theorem 1

$$PH^1(\Omega;\mathcal{P}) \cap \mathsf{W}_\varepsilon \text{ dense in } \mathsf{W}_\varepsilon \Leftrightarrow PH^2(\Omega;\mathcal{P}) \cap H_\varepsilon \text{ dense in } H_\varepsilon$$

Proof: define a linear application $\Phi : \mathsf{W}_\varepsilon \to H_\varepsilon$ by $\Phi(\boldsymbol{E}) = p$ with

$$\begin{cases} \mathbf{grad}\, p = \boldsymbol{E} - \boldsymbol{E}_R \\ l(p) = 0 \end{cases}$$

and show that Φ is continous and onto. □

Norm for H_ε:

$$|u|_{H_\varepsilon} = \left(\|\Delta_\varepsilon u\|^2_{0,\Omega} + \sum_{F \in \mathcal{F}_{ext}} \|\nabla_T u\|^2_{0,F} \right)^{1/2}$$

Introduce

$$H_0 = \left\{ u \in H^1(\Omega) \,\middle|\, \Delta_\varepsilon u \in L^2(\Omega);\ u_{|F} \in H^1_0(F)\ \forall F \in \mathcal{F}_{ext} \right\}$$

$\Rightarrow$

$$H_0 \hookrightarrow H_\varepsilon$$

Write

$$H_\varepsilon = \overline{PH^2(\Omega;\mathcal{P}) \cap H_\varepsilon} \oplus \mathcal{O}$$

where $\mathcal{O}$ is the orthogonal complement of $\overline{PH^2(\Omega;\mathcal{P}) \cap H_0}$ in H_0.

Dual singularities of the scalar transmission operator

$$L^2(\Omega) = \Delta_\varepsilon^{\mathsf{Dir}} \left(PH^2(\Omega;\mathcal{P}) \cap \mathcal{D}(\Delta_\varepsilon^{\mathsf{Dir}}) \right) \overset{\perp}{\oplus} \mathcal{N}_{\varepsilon,Dir}$$

Characterization of $\mathcal{N}_{\varepsilon,Dir}$:

$$g \in \mathcal{N}_{\varepsilon,Dir} \Leftrightarrow (g, \Delta_\varepsilon u)_{0,\Omega} = 0 \; \forall u \in H_{\varepsilon,0,\mathsf{reg}}$$

For any $\lambda \in \Lambda_{\varepsilon,S}^{\mathsf{Dir}} \cap]0,1[$, set

$$g_{S,\lambda}(r,\theta) = \eta_S(r) r^{-\lambda} \Phi_\lambda(\theta) + h_{S,\lambda}(r,\theta).$$

Technical assumption: $1 \notin \Lambda_{\varepsilon,S}^{\mathsf{Dir}}$.

$$\Rightarrow \quad \bigcup_{\mathcal{S}_{ext} \cup \mathcal{S}_{int}} \left\{ g_{S,\lambda} \,\middle|\, \lambda \in \Lambda_{\varepsilon,S}^{\mathsf{Dir}} \cap]0,1[\right\} \text{ is a basis of } \mathcal{N}_{\varepsilon,Dir}$$

Proposition

1. For any $f \in \mathcal{O}$ there is a unique $g \in \mathcal{N}_{\varepsilon,Dir}$ such that

$$\left| \begin{array}{l} \Delta_\varepsilon f = g \text{ in } \Omega \\ \Delta_T f = -\varepsilon \partial_n g \text{ in } H^{-1}(F)\ \forall F \in \mathcal{F}_{ext} \\ |f|_{H_\varepsilon} \leq \|g\|_{0,\Omega} + \sum_{F \in \mathcal{F}_{ext}} \|\varepsilon \partial_n g\|_{-1,F} \end{array} \right. \qquad (1)$$

2. For any $g \in \mathcal{N}_{\varepsilon,Dir}$ such that

$$\varepsilon \partial_n g \in H^{-1}(F)\ \forall F \in \mathcal{F}_{ext}$$

there is a unique $f \in \mathcal{O}$ satisfying (1).

Is $\varepsilon\partial_n g \in H^{-1}(F)$ possible for $g \in \mathcal{N}_{\varepsilon,Dir}$?

Let $g \in \mathcal{N}_{\varepsilon,Dir}$:

$$g = \sum_S \sum_{\lambda \in \Lambda^{\mathsf{Dir}}_{\varepsilon,S} \cap]0,1[} c_{S,\lambda} g_{S,\lambda}$$

and

$$\partial_n g_{S,\lambda} = r^{-\lambda-1}\Phi'_\lambda + h$$

Let $S \in \mathcal{S}_{ext}$:

$$\varepsilon\partial_n g \in H^{-1}(F) \quad \Rightarrow \quad c_{S,\lambda} = 0 \quad \forall \lambda \in \Lambda^{\mathsf{Dir}}_{\varepsilon,S} \cap [\tfrac{1}{2}, 1[$$

Theorem 2

Two situations may occur:

- $\Lambda^{\mathsf{Dir}}_{\varepsilon,S} \cap]0,\frac{1}{2}[= \emptyset \ \forall S \in \mathcal{S}_{ext}$ and $\Lambda^{\mathsf{Dir}}_{\varepsilon,S} \cap]0,1[= \emptyset \ \forall S \in \mathcal{S}_{int}$

$\Rightarrow g \equiv 0$

$\Rightarrow PH^2(\Omega;\mathcal{P}) \cap H_\varepsilon$ is dense in H_ε

- The above condition is violated.

$\Rightarrow$
$$H_\varepsilon = \overline{PH^2(\Omega;\mathcal{P}) \cap H_\varepsilon} \oplus \mathsf{Span}\, B$$

where

$$\begin{aligned} B \quad = \quad & \Big\{ S_{S,\lambda} \,\Big|\, S \in \mathcal{S}_{ext} \text{ and } \lambda \in \Lambda^{\mathsf{Dir}}_{\varepsilon,S} \cap]0,\tfrac{1}{2}[\Big\} \\ & \cup \Big\{ S_{S,\lambda} \,\Big|\, S \in \mathcal{S}_{int} \text{ and } \lambda \in \Lambda^{\mathsf{Dir}}_{\varepsilon,S} \cap]0,1[\Big\} \end{aligned}$$

A glance at the 3D case

- For a vertex S, introduce the set of positive singular exponents

$$\Lambda^{\mathsf{Dir}}_{\varepsilon,S} = \left\{ -\tfrac{1}{2} + \sqrt{\nu_j + \tfrac{1}{4}} \mid j \geq 1, \nu_j > 0 \right\},$$

  where $(\nu_j)_j$ are the eigenvalues of the Laplace-Beltrami operator acting on functions defined on the intersection of Ω with the unit sphere.

- For an edge e, introduce the "cross-section" $\Omega_e \subset \mathbb{R}^2$ such that the dihedral cone

$$D_e = \Omega_e \times \mathbb{R}$$

  coincides with Ω for any $z \in]-h, h[$. If S_e is the vertex of Ω_e corresponding to the edge e, set

$$\Lambda^{\mathsf{Dir}}_{\varepsilon,e} = \Lambda^{\mathsf{Dir}}_{\varepsilon,S_e}.$$

Theorem 3 (density result)

Technical assumption: $\forall S \in \mathcal{S},\ \frac{1}{2} \notin \Lambda^{\mathsf{Dir}}_{\varepsilon,S}$ and $\forall e \in \mathcal{E},\ 1 \notin \Lambda^{\mathsf{Dir}}_{\varepsilon,e}$.

The space $PH^2(\Omega;\mathcal{P}) \cap H_\varepsilon$ is dense in H_ε if and only if

$$\Lambda^{\mathsf{Dir}}_{\varepsilon,S} \cap]0,\tfrac{1}{2}[= \emptyset,\ \forall S \in \mathcal{S}_{int} \tag{2}$$

as well as

$$\Lambda^{\mathsf{Dir}}_{\varepsilon,e} \cap]0,\tfrac{1}{2}[= \emptyset\, \forall S \in \mathcal{E}_{ext} \text{ and } \Lambda^{\mathsf{Dir}}_{\varepsilon,e} \cap]0,1[= \emptyset\, \forall S \in \mathcal{E}_{int} \tag{3}$$

hold.

- If there is an exterior edge e and λ such that

$$\lambda \in \Lambda^{\mathsf{Dir}}_{\varepsilon,e} \cap]0, \tfrac{1}{2}[,$$

introduce the edge singularity

$$\mathcal{S}_{e,\lambda} = \chi(z) S_{S,\lambda}$$

and show that

$$\mathcal{S}_{e,\lambda} \notin \overline{PH^2(\Omega; \mathcal{P}) \cap H_\varepsilon}.$$

- If there is an interior vertex S and λ such that

$$\lambda \in \Lambda^{\mathsf{Dir}}_{\varepsilon,S} \cap]0, \tfrac{1}{2}[,$$

the results of [Costabel/Dauge/Nicaise, '99] apply.

Remarks

- In the homogeneous case (ε constant), condition (3) is always fulfilled.
- In 2D: if $S \in \mathcal{S}_{ext}$ belongs to at least three sub-domains, coefficients ε_j may be found such that $\Lambda^{\mathsf{Dir}}_{\varepsilon,S} \cap]0, \frac{1}{2}[$ is not empty.[a]

Conclusion

$\Rightarrow$ Extension of density results for homogeneous body

$\Rightarrow$ "Hybrid" situation depending on the electromagnetic coefficients

[a] see [Costabel/Dauge/Nicaise, '99]