Space-Time Regularity of the Solution to Maxwell's Equations in Singular Domains

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Outline

1. Basic results, for all Lipschitz domains
   1.1. Semi-group approach for the first-order Maxwell system
   1.2. Variational approach for the second-order Maxwell equations
   1.3. Application of space decomposition

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First-order formulation

Evolution equations in $\Omega \times]0, T[$:

$$\frac{\partial \mathbf{E}}{\partial t} - c^2 \,\mathbf{curl}\, \mathbf{B} = -\frac{\mathbf{J}}{\varepsilon_0}, \qquad \frac{\partial \mathbf{B}}{\partial t} + \mathbf{curl}\, \mathbf{E} = 0. \quad (1)$$

Constraint equations:

$$\operatorname{div} \mathbf{E} = \frac{\varrho}{\varepsilon_0}, \qquad \operatorname{div} \mathbf{B} = 0 \qquad \text{in } \Omega \times]0, T[\,; \quad (2)$$

$$\mathbf{E} \times \mathbf{n} = 0, \qquad \mathbf{B} \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma \times]0, T[\,. \quad (3)$$

Initial conditions

$$\mathbf{E}(0) = \mathbf{E}_0, \;\; \mathbf{B}(0) = \mathbf{B}_0 \qquad \text{in } \Omega, \qquad (4)$$

To be consistent:

$$\operatorname{div} \mathbf{E}_0 = \frac{\varrho(0)}{\varepsilon_0}, \qquad \operatorname{div} \mathbf{B}_0 = 0 \qquad \text{in } \Omega; \qquad (5)$$

$$\mathbf{E}_0 \times \mathbf{n} = 0, \qquad \mathbf{B}_0 \cdot \mathbf{n} = 0 \qquad \text{on } \Gamma. \qquad (6)$$

Charge conservation equation:

$$\operatorname{div} \mathbf{J} + \frac{\partial \varrho}{\partial t} = 0 \qquad \text{in } \Omega \times]0, T[\,. \qquad (7)$$

Semi-group approach

Evolution operator associated to (1):

$$\mathsf{A} = \begin{pmatrix} 0 & -c\,\mathbf{curl} \\ c\,\mathbf{curl} & 0 \end{pmatrix}.$$

Pivot space: $\mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega)$ [Finite energy.]
Domain: $D(\mathsf{A}) = \mathbf{H}_0(\mathbf{curl}\,;\Omega) \times \mathbf{H}(\mathbf{curl}\,;\Omega)$
dense in pivot space.

A is closed and skew-self-adjoint:
$\Rightarrow$ generates semi-group. [Stone's thm]

Theorem 1 *If* $\mathbf{J} \in \mathcal{C}^1\left(0, T; \mathbf{L}^2(\Omega)\right)$ *then (1,4) has a unique solution*

$$\begin{aligned} (\mathbf{E}, \mathbf{B}) \;&\in\; \mathcal{C}^0\left(0, T; \mathbf{H}_0(\mathbf{curl}\,;\Omega) \times \mathbf{H}(\mathbf{curl}\,;\Omega)\right) \\ &\cap\; \mathcal{C}^1\left(0, T; \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega)\right) \end{aligned}$$

If (5–6) and (7) hold, with $\varrho \in \mathcal{C}^0\left(0, T; L^2(\Omega)\right)$*, then the constraints (2–3) hold* $\forall t$ *and:*

$$(\mathbf{E}, \mathbf{B}) \in \mathcal{C}^0\left(0, T; \mathbf{X} \times \mathbf{Y}\right)$$

with:

$$\mathbf{X} := \mathbf{H}_0(\mathbf{curl}\,;\Omega) \cap \mathbf{H}(\mathrm{div}\,;\Omega),$$
$$\mathbf{Y} := \mathbf{H}(\mathbf{curl}\,;\Omega) \cap \mathbf{H}_0(\mathrm{div}\,;\Omega).$$

Second-order formulation

– Used in numerical methods
– Allows to almost decouple $\mathbf{E}$ and $\mathbf{B}$

Electric field equations:

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} + c^2\, \mathbf{curl}\, \mathbf{curl}\, \mathbf{E} = -\frac{\mathbf{J}}{\varepsilon_0} \quad \text{in } \Omega \times]0,T[\,; \tag{8}$$

$$\mathbf{E}(0) = \mathbf{E}_0, \quad \text{in } \Omega; \tag{9}$$

$$\frac{\partial \mathbf{E}}{\partial t}(0) = c^2\, \mathbf{curl}\, \mathbf{B}_0 - \frac{\mathbf{J}(0)}{\varepsilon_0} \quad \text{in } \Omega; \tag{10}$$

$$\operatorname{div} \mathbf{E} = \frac{\varrho}{\varepsilon_0} \quad \text{in } \Omega \times]0,T[\,, \tag{11}$$

$$\mathbf{E} \times \mathbf{n} = 0 \quad \text{on } \Gamma \times]0,T[\,. \tag{12}$$

+ consistency conditions
+ charge conservation

Variational approach (*à la* Lions) for the second-order electric field equations

Pivot space: $\mathbf{L}^2(\Omega)$; "Energy" space: $\mathbf{H}_0(\mathbf{curl}\,;\Omega)$

Theorem 2 *If* $\mathbf{J} \in H^1\left(0,T;\mathbf{L}^2(\Omega)\right)$ *then (8–10) has a unique solution*

$$\mathbf{E} \in \mathcal{C}^0\left(0,T;\mathbf{H}_0(\mathbf{curl}\,;\Omega)\right) \cap \mathcal{C}^1\left(0,T;\mathbf{L}^2(\Omega)\right).$$

If consistency and conservation conditions hold, then (11) holds $\forall t$ *and*

$$\varrho \in \mathcal{C}^0\left(0,T;L^2(\Omega)\right) \Rightarrow \mathbf{E} \in \mathcal{C}^0\left(0,T;\mathbf{X}\right).$$

Second-order formulation, Magnetic field

$$\frac{\partial^2 \mathbf{B}}{\partial t^2} + c^2 \,\mathbf{curl}\,\mathbf{curl}\,\mathbf{B} = \frac{1}{\varepsilon_0}\,\mathbf{curl}\,\mathbf{J} \quad \text{in } \Omega \times]0,T[\,; \quad (13)$$

$$\mathbf{B}(0) = \mathbf{B}_0, \quad \frac{\partial \mathbf{B}}{\partial t}(0) = -\mathbf{curl}\,\mathbf{E}_0 \text{ in } \Omega; \quad (14)$$

$$\operatorname{div}\mathbf{B} = 0 \quad \text{in } \Omega \times]0,T[\,, \quad (15)$$

$$\mathbf{B}\cdot\mathbf{n} = 0 \quad \text{on } \Gamma \times]0,T[\,. \quad (16)$$

$$(\varepsilon_0 c^2\,\mathbf{curl}\,\mathbf{B} - \mathbf{J}) \times \mathbf{n} = 0 \text{ on } \Gamma \times]0,T[\,. \quad (17)$$

$+$ consistency conditions

Magnetic field equations: Variational theory

Pivot space: $\mathbf{L}^2(\Omega)$; "Energy" space: $\mathbf{H}(\mathbf{curl}\,;\Omega)$

Theorem 3 *If* $\mathbf{J} \in L^2(0,T;\mathbf{H}(\mathbf{curl}\,;\Omega))$ *then (13–14) has a unique solution*

$$\mathbf{B} \in \mathcal{C}^0(0,T;\mathbf{H}(\mathbf{curl}\,;\Omega)) \cap \mathcal{C}^1\left(0,T;\mathbf{L}^2(\Omega)\right).$$

If consistency conditions hold, then (15–17) hold $\forall t$ *and* $\mathbf{B} \in \mathcal{C}^0(0,T;\mathbf{Y})$.

Metaphysical Question:
Requirement $\mathbf{J} \in \mathbf{H}(\mathbf{curl}\,;\Omega)$ unphysical.

Pragmatic Answer:
– Reasoning: Couple second-order formulation for $\mathbf{E}$ with first-order for $\mathbf{B}$
– Numerical computations with two second-order formulations work well.

A Pretty Commonplace

If Ω has no reentrant singularities, then

$$\mathbf{X} \text{ and } \mathbf{Y} \subset \mathbf{H}^1(\Omega)$$

hence

$$\mathbf{E},\ \mathbf{B} \in \mathcal{C}^0\left(0, T; \mathbf{H}^1(\Omega)\right) \cap \mathcal{C}^1\left(0, T; \mathbf{H}^0(\Omega)\right).$$

The results of Thms 1, 2, 3 are optimal on the scale $\mathcal{C}^\alpha(0, T; \mathbf{H}^s(\Omega))$.

In presence of non-convex singularities:
the regular subspaces

$$\mathbf{X}_R := \mathbf{X} \cap \mathbf{H}^1(\Omega), \qquad \mathbf{Y}_R := \mathbf{Y} \cap \mathbf{H}^1(\Omega),$$

are **closed** within the natural spaces.
We introduce the splittings

$$\mathbf{X} := \mathbf{X}_R \oplus \mathbf{X}_S, \qquad \mathbf{E}(t) := \mathbf{E}_R(t) + \mathbf{E}_S(t); \tag{18}$$

$$\mathbf{Y} := \mathbf{Y}_R \oplus \mathbf{Y}_S, \qquad \mathbf{B}(t) := \mathbf{B}_R(t) + \mathbf{B}_S(t). \tag{19}$$

A Pretty Commonplace, Cont'd

Theorem 4 *As the projection onto closed subspaces is smooth, there holds:*

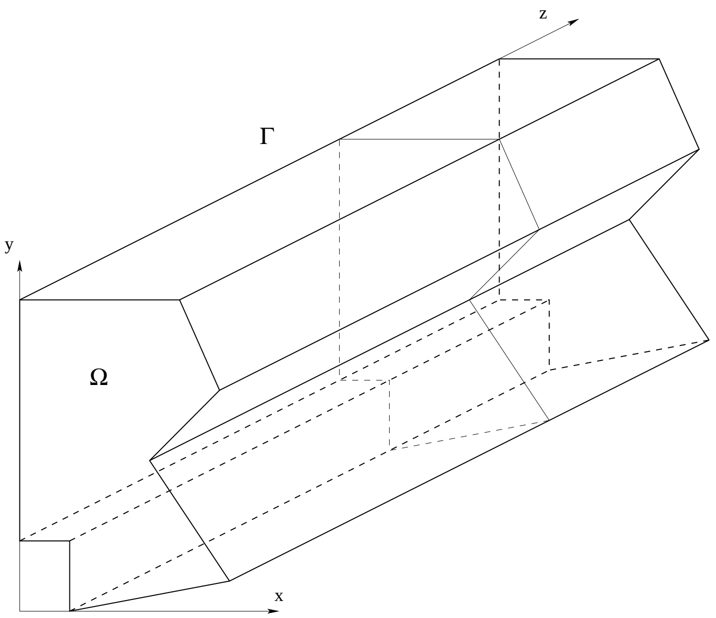
$$(\mathbf{E}_R, \mathbf{E}_S) \in \mathcal{C}^0(0, T; \mathbf{X}_R \times \mathbf{X}_S),$$
$$(\mathbf{B}_R, \mathbf{E}_S) \in \mathcal{C}^0(0, T; \mathbf{Y}_R \times \mathbf{Y}_S).$$

Result of Thm 4 powerful: indepedent of the structure of singular subspaces. Orthogonality, etc. Generally not optimal : does not exploit it.

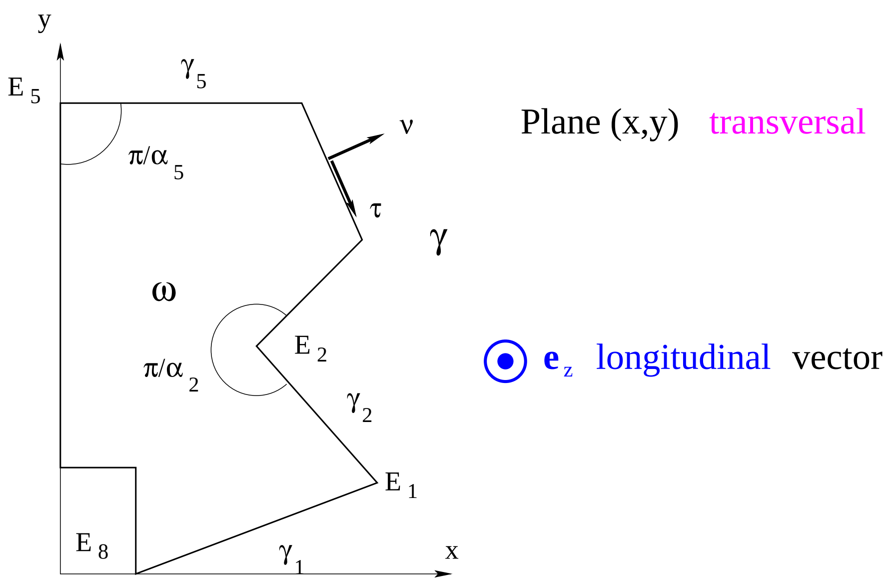
When this structure is better known, more precise results can be proven.

Example: in the case of dimension reduction.

‘Plane’ Cartesian Geometry



Ω invariant by translation along z;
ω polygon with boundary γ.



Invariance by Translation and Operators

The **transversal** and **longitudinal** components of the field $\mathbf{u}$ are defined as:

$$\mathbf{u}_\perp := u_x\,\mathbf{e}_x + u_y\,\mathbf{e}_y, \qquad \mathbf{u}_{||} := u_z\,\mathbf{e}_z.$$

In the (x, y) plane we define the operators

$$\mathrm{div}\,\mathbf{v} := \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}, \qquad \mathrm{rot}\,\mathbf{v} := \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y},$$

$$\mathbf{curl}\,f := \frac{\partial f}{\partial y}\,\mathbf{e}_x - \frac{\partial f}{\partial x}\,\mathbf{e}_y.$$

If $\mathbf{u}$ is invariant by translation $(\partial_z \mathbf{u} = 0)$:

$$\mathrm{div}_{3D}\,\mathbf{u} = \mathrm{div}\,\mathbf{u}_\perp, \;\; \mathbf{curl}_{3D}\,\mathbf{u} = \mathbf{curl}\,u_z + (\mathrm{rot}\,\mathbf{u}_\perp)\,\mathbf{e}_z.$$

Dimension Reduction in Plane Geometry

(i) Transverse electric (**TE**) mode:

$$\frac{\partial \mathbf{E}_\perp}{\partial t} - c^2 \,\mathbf{curl}\, B_z = -\frac{\mathbf{J}_\perp}{\varepsilon_0} \qquad \text{in } \omega \times]0,T[\,, \tag{20}$$

$$\frac{\partial B_z}{\partial t} + \mathrm{rot}\,\mathbf{E}_\perp = 0 \qquad \text{in } \omega \times]0,T[\,, \tag{21}$$

$$\mathbf{E}_\perp(0) = \mathbf{E}_{0\perp},\ \ B_z(0) = B_{0z} \qquad \text{in } \omega, \tag{22}$$

$$\mathrm{div}\,\mathbf{E}_\perp = \frac{\varrho}{\varepsilon_0} \qquad \text{in } \omega \times]0,T[\,, \tag{23}$$

$$\mathbf{E}_\perp \cdot \boldsymbol{\tau} = 0 \qquad \text{on } \gamma \times]0,T[\,, \tag{24}$$

$$\mathrm{div}\,\mathbf{J}_\perp + \frac{\partial \varrho}{\partial t} = 0 \qquad \text{in } \omega \times]0,T[\,. \tag{25}$$

Consistency:

$$\mathrm{div}\,\mathbf{E}_{0\perp} = \frac{\varrho(0)}{\varepsilon_0} \text{ in } \omega, \qquad \mathbf{E}_{0\perp} \cdot \boldsymbol{\tau} = 0 \text{ on } \gamma.$$

Dimension Reduction in Plane Geometry (cont'd)

(ii) Transverse magnetic (**TM**) mode:

$$\frac{\partial \mathbf{B}_\perp}{\partial t} + \mathbf{curl}\, E_z = 0 \qquad \text{in } \omega \times]0,T[\,, \quad (26)$$

$$\frac{\partial E_z}{\partial t} - c^2 \,\mathrm{rot}\, \mathbf{B}_\perp = -\frac{J_z}{\varepsilon_0} \qquad \text{in } \omega \times]0,T[\,, \quad (27)$$

$$\mathbf{B}_\perp(0) = \mathbf{B}_{0\perp}, \;\; E_z(0) = E_{0z} \qquad \text{in } \omega, \quad (28)$$

$$\mathrm{div}\, \mathbf{B}_\perp = 0 \qquad \text{in } \omega \times]0,T[\,, \quad (29)$$

$$E_z = 0, \quad \mathbf{B}_\perp \cdot \boldsymbol{\nu} = 0 \qquad \text{on } \gamma \times]0,T[\,. \quad (30)$$

Consistency:

$$\mathrm{div}\, \mathbf{B}_{0\perp} = 0 \text{ in } \omega, \;\; E_{0z} = \mathbf{B}_{0\perp} \cdot \boldsymbol{\nu} = 0 \text{ on } \gamma.$$

Both modes can be rewritten as almost decoupled vector and scalar wave equations.

Basic Results for Plane Geometry

Here: $\mathbf{X} := \mathbf{H}_0(\mathrm{rot}\,;\omega) \cap \mathbf{H}(\mathrm{div}\,;\omega)$,

$\mathbf{Y} := \mathbf{H}(\mathrm{rot}\,;\omega) \cap \mathbf{H}_0(\mathrm{div}\,;\omega)$.

[Infinite length $\Rightarrow$ energy only locally finite]

With these modifications, Thms 1, 2, 3 say:

$$\mathbf{E}_\perp \in \mathcal{C}^0(0,T;\mathbf{X}) \cap \mathcal{C}^1\left(0,T;\mathbf{L}^2(\omega)\right); \quad (31)$$

$$B_z \in \mathcal{C}^0\left(0,T;H^1_*(\omega)\right) \cap \mathcal{C}^1\left(0,T;L^2_*(\omega)\right); \quad (32)$$

$$\mathbf{B}_\perp \in \mathcal{C}^0(0,T;\mathbf{Y}) \cap \mathcal{C}^1\left(0,T;\mathbf{L}^2(\omega)\right); \quad (33)$$

$$E_z \in \mathcal{C}^0\left(0,T;H^1_0(\omega)\right) \cap \mathcal{C}^1\left(0,T;L^2(\omega)\right). \quad (34)$$

[$*$ means zero average.]

z-components are regular in space.

$\Rightarrow$ Focus on transversal components.

Decomposition of Transversal Components w.r.t. Regularity

$\mathbf{X}_S$ and $\mathbf{Y}_S$ are finite-dimensional:
$\dim \mathbf{X}_S = \dim \mathbf{Y}_S = \#$ reentrant corners in ω.

$$\mathbf{E}_\perp(t) = \mathbf{E}_R(t) + \sum_{i=1}^{N_C} \kappa_i^{\mathbf{E}}(t)\, \mathbf{x}_{S*}^i \quad (35)$$

$$\mathbf{B}_\perp(t) = \mathbf{B}_R(t) + \sum_{i=1}^{N_C} \kappa_i^{\mathbf{B}}(t)\, \mathbf{y}_{S*}^i \quad (36)$$

Space regularity controlled by the $\mathbf{x}_{S*}^i$:

$\mathbf{x}_{S*}^i \in \mathbf{H}^{\alpha_i-\epsilon}(\omega)$, hence $\mathbf{E}_\perp(t) \in \mathbf{H}^{\alpha_{\min}-\epsilon}(\omega)$.

This regularity is optimal.

Costabel–Dauge, Assous–Ciarlet,. . .

Time regularity controlled by $\mathbf{E}_R$ and the $\kappa_i^{\mathbf{E}}$.

Hodge Decomposition of Transversal Comp'ts

Independently of (35), we split $\mathbf{E}_\perp(t)$ as:

$$\mathbf{E}_\perp(t) = -\mathbf{grad}\, V(t) + \mathbf{curl}\, W(t). \qquad (37)$$

V, W unique provided chosen resp. within

$$\begin{aligned} \Phi &:= \left\{\phi \in H_0^1(\omega) : \Delta\phi \in L^2(\omega)\right\}, \\ \Psi &:= \left\{\psi \in H_*^1(\omega) : \Delta\psi \in L^2(\omega),\ \partial_\nu\psi|_\gamma = 0\right\}, \end{aligned}$$

'natural' spaces of potentials.

By Thms 1 and 2:

$$\begin{aligned} V &\in C^0(0,T;\Phi) \cap C^1(0,T;H_0^1(\omega)), \quad (38) \\ W &\in C^0(0,T;\Psi) \cap C^1(0,T;H_*^1(\omega)). \quad (39) \end{aligned}$$

Decomposition of Potentials w.r.t. Regularity

The regular subspaces of potentials

$$\Phi_R := \Phi \cap H^2(\omega), \quad \Psi_R := \Psi \cap H^2(\omega),$$

are closed and of codimension N_C.

$$V(t) = V_R(t) + \sum \kappa_i^V(t)\, \phi_{S*}^i, \quad (40)$$

$$W(t) = W_R(t) + \sum \kappa_i^W(t)\, \psi_{S*}^i. \quad (41)$$

We can choose the singular bases s.t.:

$$-\mathbf{grad}\, \phi_{S*}^i + \mathbf{curl}\, \psi_{S*}^i = 2\, \mathbf{x}_{S*}^i + 2\, \mathbf{w}_+^i$$

$$-\mathbf{grad}\, \phi_{S*}^i - \mathbf{curl}\, \psi_{S*}^i = 2\, \mathbf{w}_-^i, \quad \mathbf{w}_\pm^i \in \mathbf{X}_R.$$

The decompositions (35), (40), (41) are linked:

$$\kappa_i^{\mathbf{E}}(t) = \kappa_i^V(t) + \kappa_i^W(t), \quad (42)$$

$$\mathbf{E}_R(t) = \mathbf{curl}\, W_R(t) - \mathbf{grad}\, V_R(t) + \sum \kappa_i^{\mathbf{E}}(t)\, \mathbf{w}_+^i + \sum (\kappa_i^V(t) - \kappa_i^W(t))\, \mathbf{w}_-^i \quad (43)$$

Time reg'ty controlled by V_R, W_R, κ_i^V, κ_i^W.

Equations Satisfied by the Potentials

V solution of elliptic equation

$$-\Delta V(t) = \frac{\varrho(t)}{\varepsilon_0} \text{ in } \omega, \quad V(t) = 0 \text{ on } \gamma.$$

W solution of hyperbolic equation

$$\partial_t^2 W - c^2 \Delta W = \partial_t f \text{ in } \omega \times]0,T[\,, \tag{44}$$

$$\partial_\nu W = 0 \text{ on } \gamma \times]0,T[\,, \tag{45}$$

$$W(0) = W_0, \quad \partial_t W|_{t=0} = W_1 \text{ in } \omega. \tag{46}$$

where $\partial_t f$, W_0, W_1 are defined by:

$$\mathbf{curl}\, f(t) = -\mathbf{J}_\perp(t) + \partial_t \mathbf{grad}\, V(t),$$

$$\mathbf{curl}\, W_0 = \mathbf{E}_{0\perp} + \mathbf{grad}\, V(0), \; W_1 = c^2 B_{0z} + f(0).$$

W regular enough (39) to be strong solution to (44–46) if the latter exists.

Condition:
$\partial_t f \in L^1\left(0,T; H^1(\omega)\right) \iff \mathbf{J}_\perp \in W^{1,1}\left(0,T; \mathbf{L}^2(\omega)\right)$.

Space-Time Regularity for the Electric Field

Theorem 5 *Assume that the sources satisfy:*

$$\varrho \in \mathcal{C}^{0,1-\alpha_{\max}-\epsilon}\left(0,T;L^2(\omega)\right), \ \forall\epsilon>0, \quad (47)$$

$$\mathbf{J}_\perp \in W^{1,1}\left(0,T;\mathbf{L}^2(\omega)\right). \quad (48)$$

Then the following results hold $\forall\epsilon,\ \epsilon'>0$*:*

$$\kappa_i^{\mathbf{E}} \in \mathcal{C}^{0,1-\alpha_{\max}-\epsilon}(0,T;\mathrm{R}), \quad (49)$$

$$\mathbf{E}_\perp \in \mathcal{C}^{0,1-\alpha_{\max}-\epsilon}\left(0,T;\mathbf{H}^{\alpha_{\min}-\epsilon'}(\omega)\right). \quad (50)$$

- Ellipticity + smoothness of projections for V

$$(53) \Rightarrow V \in \mathcal{C}^{0,1-\alpha_{\max}-\epsilon}(0,T;\Phi)$$

$$\Rightarrow \begin{cases} V_R \in \mathcal{C}^{0,1-\alpha_{\max}-\epsilon}(0,T;\Phi_R) \\ \kappa_i^V \in \mathcal{C}^{0,1-\alpha_{\max}-\epsilon}(0,T;\mathrm{R}) \end{cases}$$

- W as solution to the scalar wave problem (44–46) satisfies [Grisvard 92]

$$W_R \in \mathcal{C}^{0,1-\alpha_{\max}-\epsilon}\left(0,T;H^{1+\alpha_{\max}+\delta}(\omega)\right),$$

$$\kappa_i^W \in \mathcal{C}^{0,1-\alpha_{\max}-\epsilon}(0,T;\mathrm{R})$$

- Then (49,50) follow from decomp'n (42–43).

Space-Time Regularity for the Electric and Magnetic Fields (end)

Theorem 6 *Under the hypotheses of Thm 5,*

$$\mathbf{E} \in \mathcal{C}^{0,1-\alpha_{\mathsf{max}}-\epsilon}\left(0,T;\mathbf{H}_{\mathsf{loc}}^{\alpha_{\mathsf{min}}-\epsilon'}(\Omega)\right). \qquad (51)$$

Similarly, if $J_z \in L^1\left(0,T;H_0^1(\omega)\right)$, *then:*

$$\mathbf{B} \in \mathcal{C}^{0,1-\alpha_{\mathsf{max}}-\epsilon}\left(0,T;\mathbf{H}_{\mathsf{loc}}^{\alpha_{\mathsf{min}}-\epsilon'}(\Omega)\right). \qquad (52)$$

Indeed, the z-components satisfy (32,34):

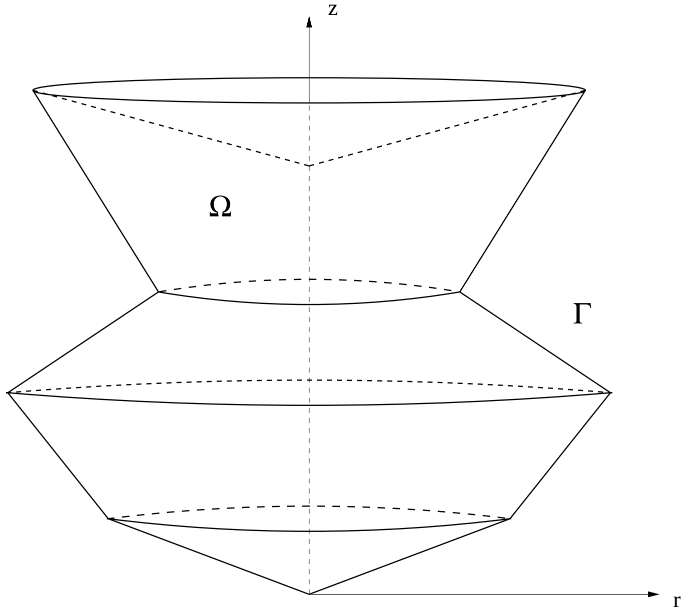
$$E_z,\ B_z \in \mathcal{C}^0\left(0,T;H^1(\omega)\right) \cap \mathcal{C}^1\left(0,T;H^0(\omega)\right),$$

hence by interpolation:

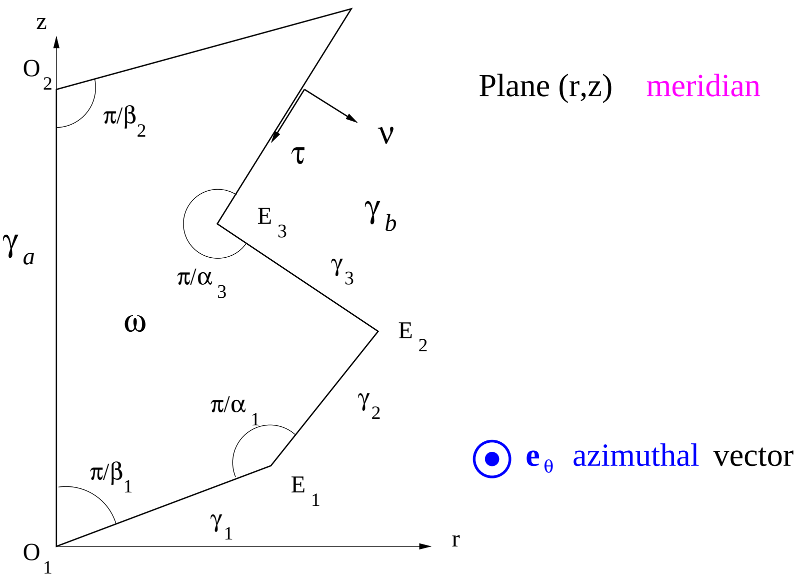
$$E_z,\ B_z \in \mathcal{C}^{0,1-\alpha_{\mathsf{max}}-\epsilon}\left(0,T;H^{\alpha_{\mathsf{max}}+\epsilon}(\omega)\right).$$

Remark:
requirement $J_z \in H_0^1(\omega)$ still unphysical.

Axisymmetric Geometry



Ω invariant by rotation around z;
ω polygon with boundary $\gamma = \gamma_a \cup \gamma_b$.



Nice Features of the Axisymmetric Geometry

Another two-dimensional situation = qualitatively similar to plane geometry.

(i) ...but often more realistic !

(ii) dimension θ finite $\Rightarrow$ energy globally finite.

$\mathbf{u}$ axisymmetric iff $\partial_\theta u_{r,\theta,z} = 0$.
Subspaces of axisymmetric fields denoted:

$$\breve{\mathbf{L}}^2(\Omega),\ \breve{\mathbf{X}},\ \breve{\mathbf{H}}^s(\Omega), \dots$$

characterised as weighted Sobolev spaces in ω.

For axisymmetric fields, the 3D **curl** and div operators decouple the **meridian** and **azimuthal** components

$$\mathbf{u}_m = u_r\,\mathbf{e}_r + u_z\,\mathbf{e}_z, \quad \mathbf{u}_\theta = u_\theta\,\mathbf{e}_\theta.$$

Nice Features of the Axisymmetric Geometry (cont'd)

$$\dim \breve{\mathbf{Y}}_S = \# \text{ reentrant circular edges in } \Omega.$$
$$\dim \breve{\mathbf{X}}_S = \textit{id.} + \# \text{ sharp vertices in } \Omega.$$

Sharp vertex: angle greater than

$$\pi/\beta_\star \simeq 130°43' \text{ s.t. } P_{1/2}(\cos \pi/\beta_\star) = 0.$$

For the i-th sharp vertex define ν_i as the unique

$$\nu \in \,]0, 1/2[\, \text{ s.t. } P_\nu(\cos \pi/\beta_i) = 0.$$

Explicit expressions known for singular bases $\mathbf{x}^i_{S*}$, $\mathbf{y}^i_{S*}$, ϕ^i_{S*}, ψ^i_{S*}. [Assous, Ciarlet, L. 02]

Singular exponents:

Type	Primal	Dual
edge	α_i	$-\alpha_i$
vertex	ν_i	$-1-\nu_i$

Space-Time Regularity for the Electric and Magnetic Fields (Axisymmetric Case)

Optimal space regularity:

$$\breve{\mathbf{X}} \subset \breve{\mathbf{H}}^{\sigma_{\min}-\epsilon}(\Omega), \quad \breve{\mathbf{Y}} \subset \breve{\mathbf{H}}^{\alpha_{\min}-\epsilon}(\Omega),$$

where

$$\begin{pmatrix} \sigma_{\min} \\ \sigma_{\max} \end{pmatrix} := \begin{pmatrix} \min \\ \max \end{pmatrix} \left\{ \begin{array}{c} \alpha_i \text{ (reentrant edges)}; \\ \nu_i + 1/2 \text{ (sharp vertices)} \end{array} \right\}.$$

Theorem 7 *Under the hypotheses*

$$\varrho \in \mathcal{C}^{0,1-\sigma_{\max}-\epsilon}\left(0,T;\breve{L}^2(\Omega)\right), \ \forall \epsilon > 0, \quad (53)$$

$$\mathbf{J}_m \in W^{1,1}\left(0,T;\breve{\mathbf{L}}^2(\Omega)\right), \quad (54)$$

the following results hold $\forall \epsilon,\ \epsilon' > 0$*:*

$$\mathbf{E} \in \mathcal{C}^{0,1-\sigma_{\max}-\epsilon}\left(0,T;\breve{\mathbf{H}}^{\sigma_{\min}-\epsilon'}(\Omega)\right). \quad (55)$$

Similarly, if $J_\theta \in L^1\left(0,T;\breve{H}^1_0(\Omega)\right)$*, then:*

$$\mathbf{B} \in \mathcal{C}^{0,1-\alpha_{\max}-\epsilon}\left(0,T;\breve{\mathbf{H}}^{\alpha_{\min}-\epsilon'}(\Omega)\right). \quad (56)$$

Conclusion

- ‘Basic’ results, completed with space decomposition, are sufficient for numerical appications.

- Two-dimensional results are finer:
  may provide some compacity for non-linear problems.

- But they depend crucially on dimension 2:
  - Finite dimension of singular spaces
  - Simplified Hodge decomposition
  - Precise results of Grisvard (92).

Now, if someone wants to attack the 3D case...